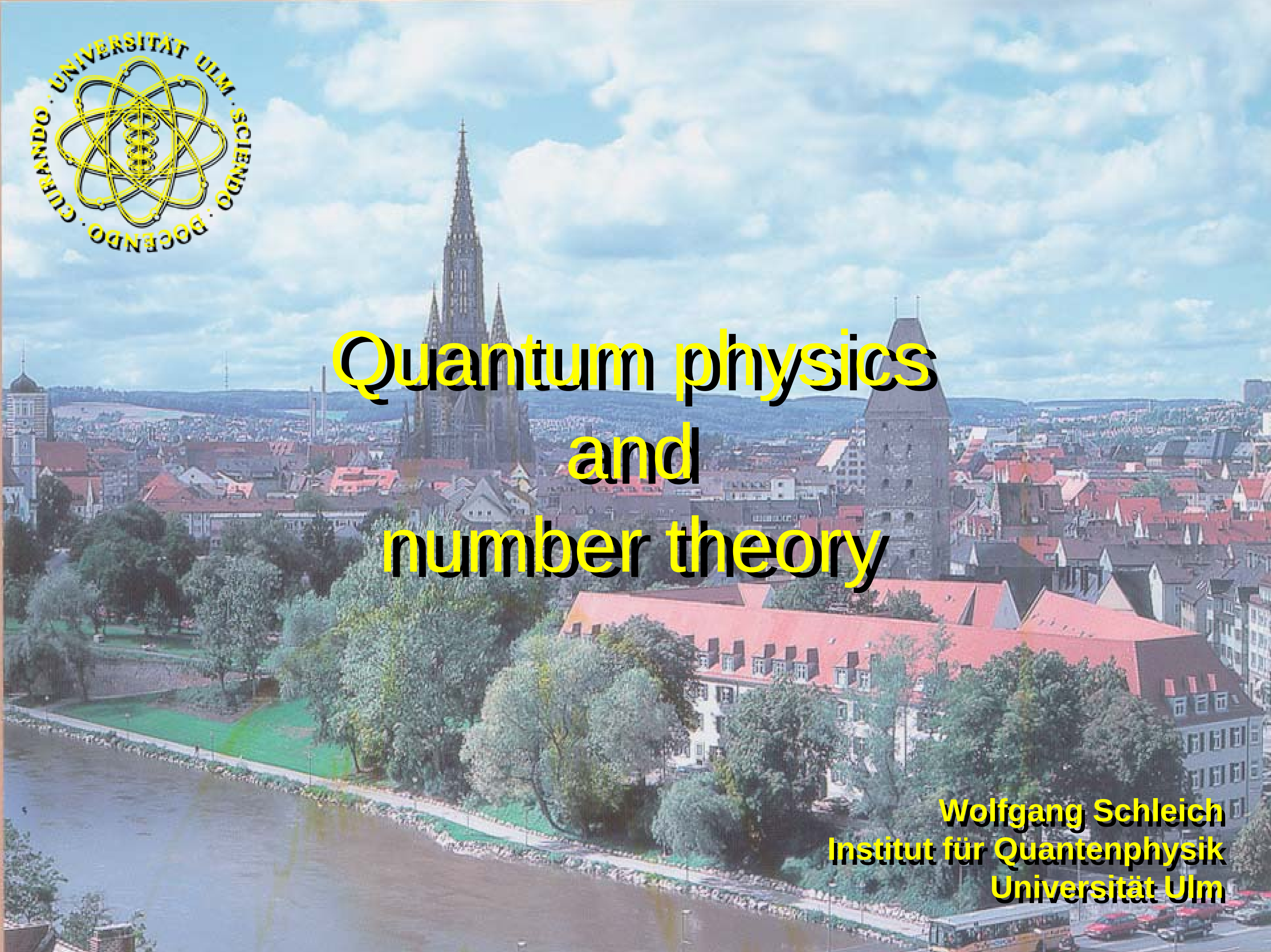

Quantum physics and number theory

Wolfgang Schleich
Institut für Quantenphysik
Universität Ulm

Overview

- Factorization with Gauss sums
    - Scaling laws
    - Implementation in atoms
    - NMR experiment
- Wave packets and Riemann Zeta function
    - Autocorrelation function
    - Logarithmic spectrum
    - Schrödinger cats

I. Sh. Averbukh	Rehovot
M. Bienert	Ulm
O. Crasser	
M. Freyberger	
B. Girard	Toulouse
D. Haase	
F. Haug	
E. Lutz	
H. Mack	
H. Maier	
M. Mehring	Stuttgart
W. Merkel	
G. G. Paulus	Texas
W. P. Schleich	

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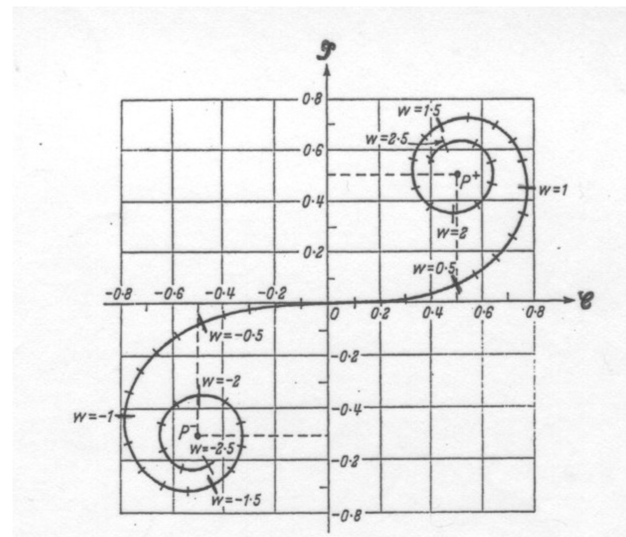
Cornu spiral

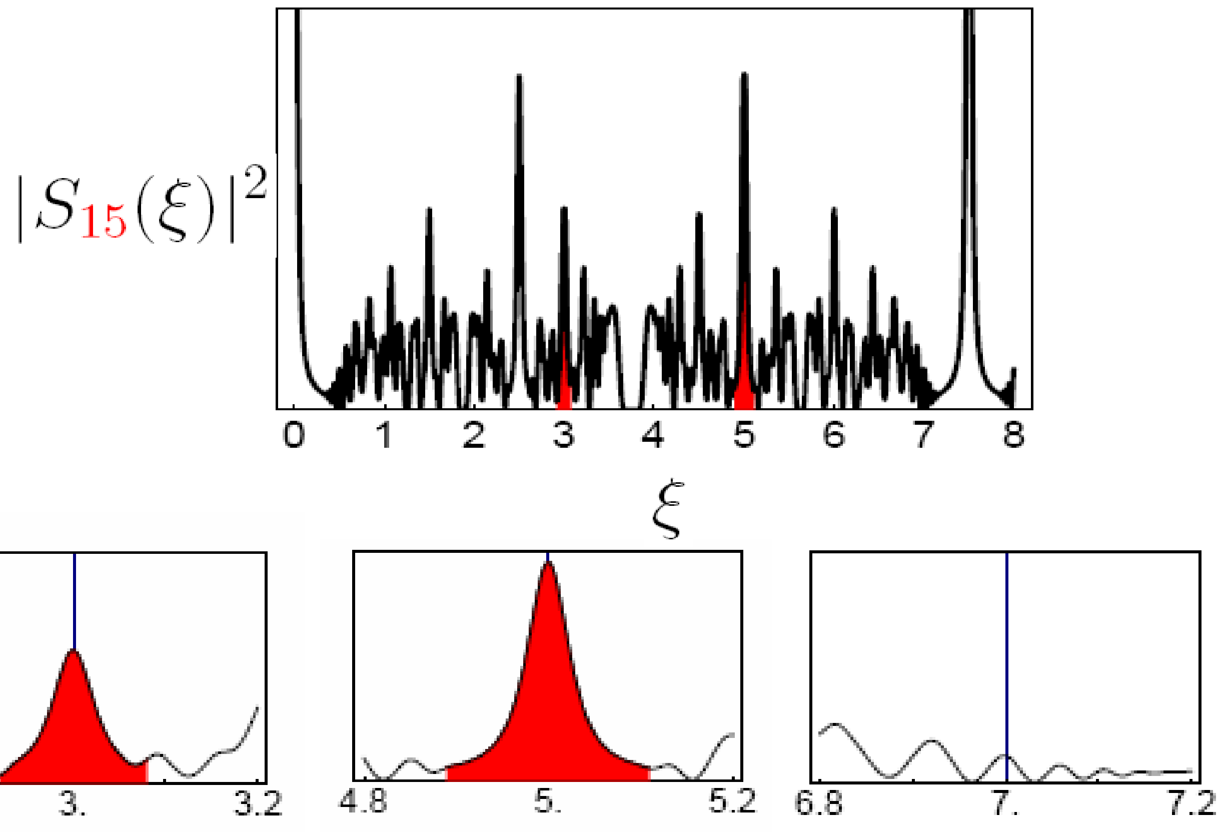
$$C(w) + iS(w) = \int_0^w d\tau \, e^{i\pi\tau^2/2}$$



Factorization with Gauss sums

$$S_N(\xi) = \sum_m w_m \exp\left[2\pi i \left(m + \frac{m^2}{N}\right)\xi\right]$$

$$\xi = \ell = q\frac{N}{r}$$



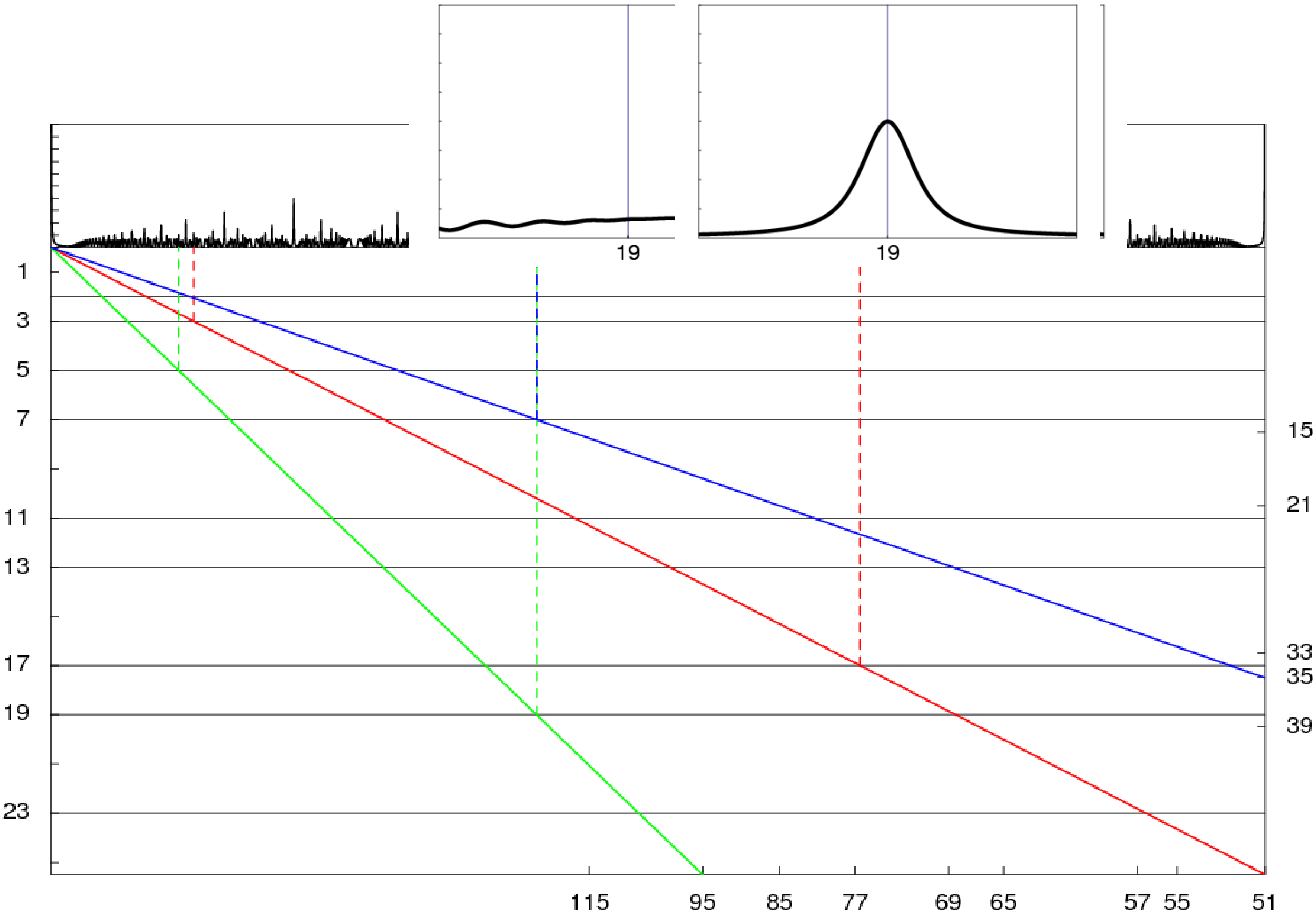
Scaling law

$$S_N(\xi) = \sum_m w_m \exp\left[2\pi i (m + \frac{m^2}{N})\xi\right]$$

$$\ell = q\frac{N}{r} = q\frac{N}{N'}\frac{N'}{r}$$

$$\ell' \equiv \frac{N'}{N}\ell = q\frac{N'}{r}$$

Rescaling



19
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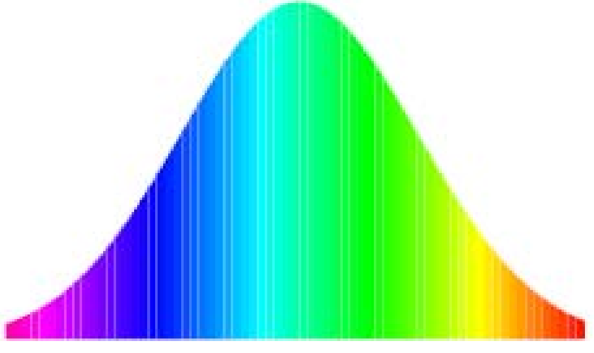
51

Chirped pulse



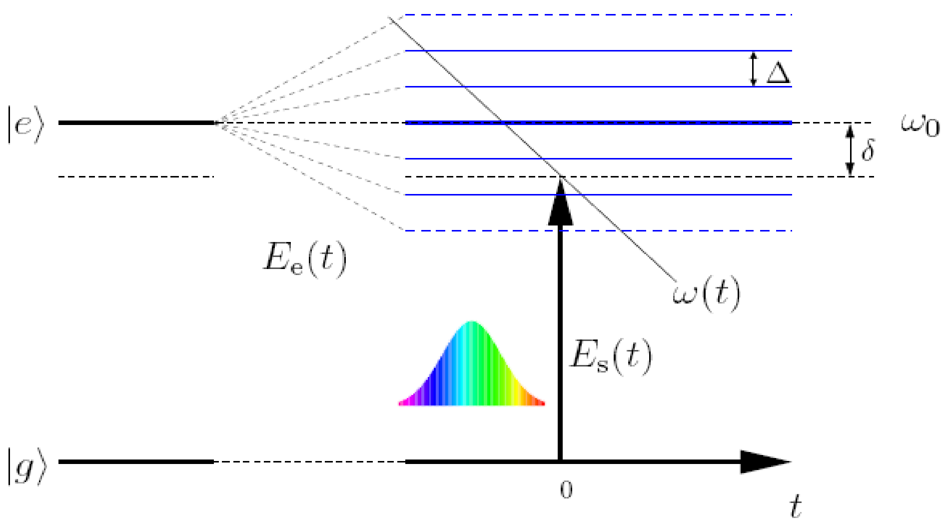
$$E(t) = \mathcal{E}_0 \left[e^{-i\omega_0 t} f(t) + \text{c.c.} \right]$$

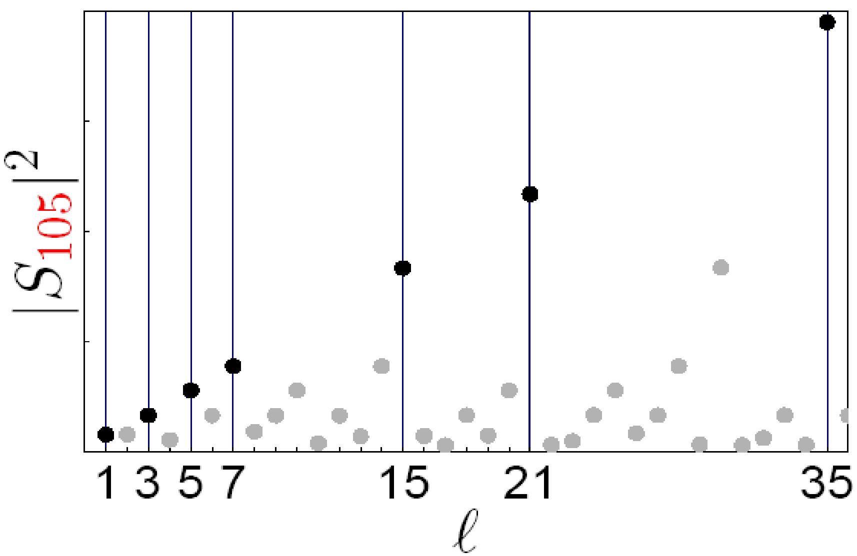
$$f(t) \propto \exp\left[-\frac{t^2}{2\sigma_r^2} \right] \exp\left[-i\frac{t^2}{2\sigma_i^2} \right]$$

$$\begin{aligned} \sigma_r &\sim \sigma_r(\phi'') \\ \sigma_i &\sim \sigma_i(\phi'') \end{aligned}$$

Floquet ladder

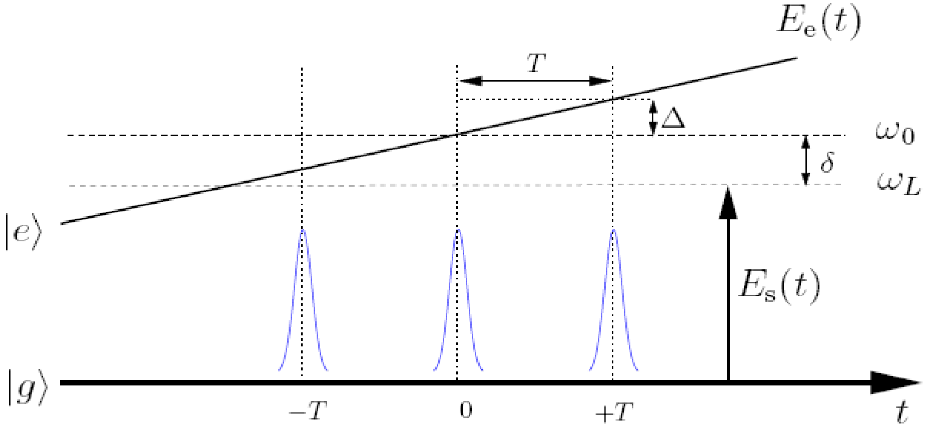
$$c_e \sim \sum_m w_m \exp\left[-2\pi i \left(m - \frac{m^2}{N}\right)\xi\right] \equiv S_N(\xi)$$

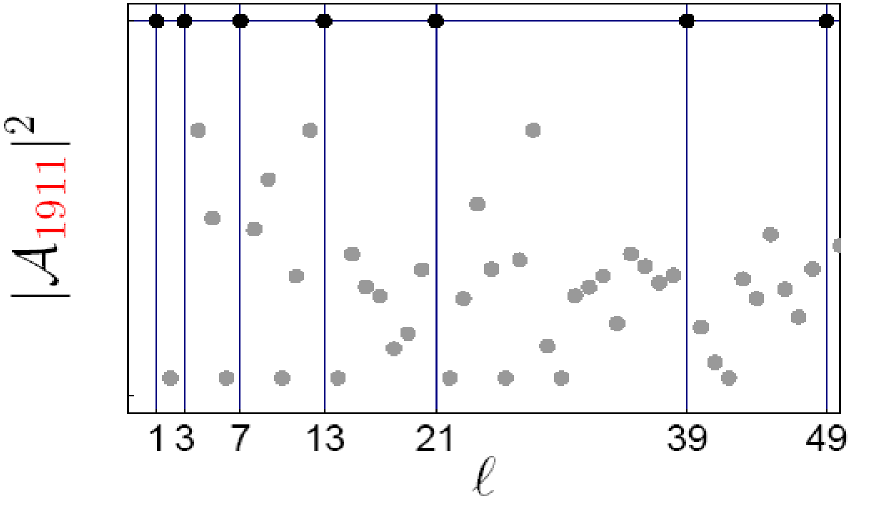




Pulse train

$$c_e \sim \sum_m \exp\left[-2\pi i\, m^2 \frac{N}{\ell}\right] \equiv \mathcal{A}_N(\ell)$$





Gauss as a schoolboy

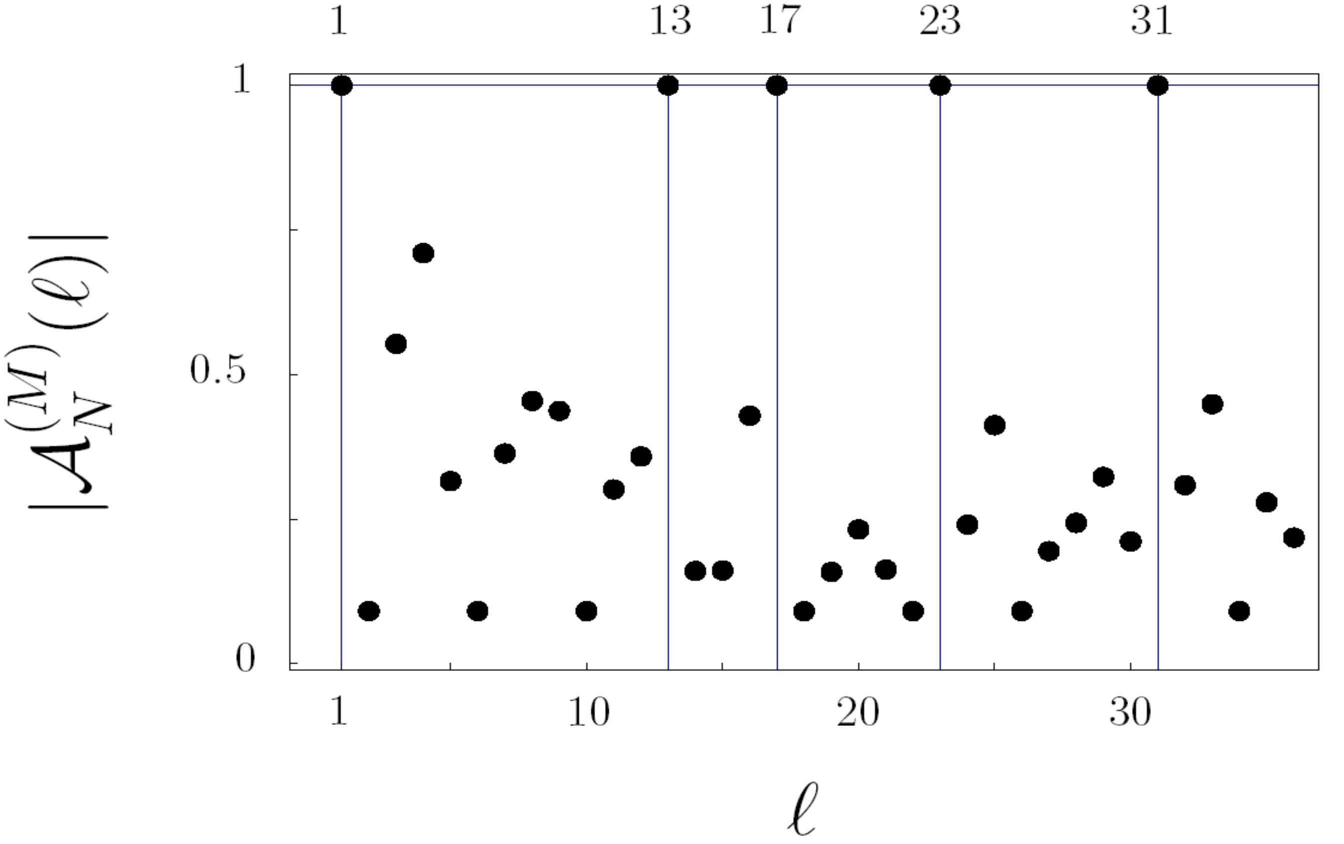
$$
\begin{array}{rcccccccccccccc}
\mathcal{S} & \equiv & 0 & + & 1 & + & 2 & + & 3 & + & \dots & \dots & + & 99 & + & 100 \\
+\,\mathcal{S} & & 100 & + & 99 & + & 98 & + & 97 & + & \dots & \dots & + & 1 & + & 0 \\
\hline
2\,\mathcal{S} & = & \multicolumn{14}{l}{\underbrace{100 + 100 + 100 + 100 + \dots \qquad \dots + 100 + 100}_{101\ \text{terms}}}
\end{array}
$$

$$\mathcal{S} = \sum_{k=0}^{m} k = \frac{1}{2} m(m+1)$$

Factorization N=157573=13·17·23·31

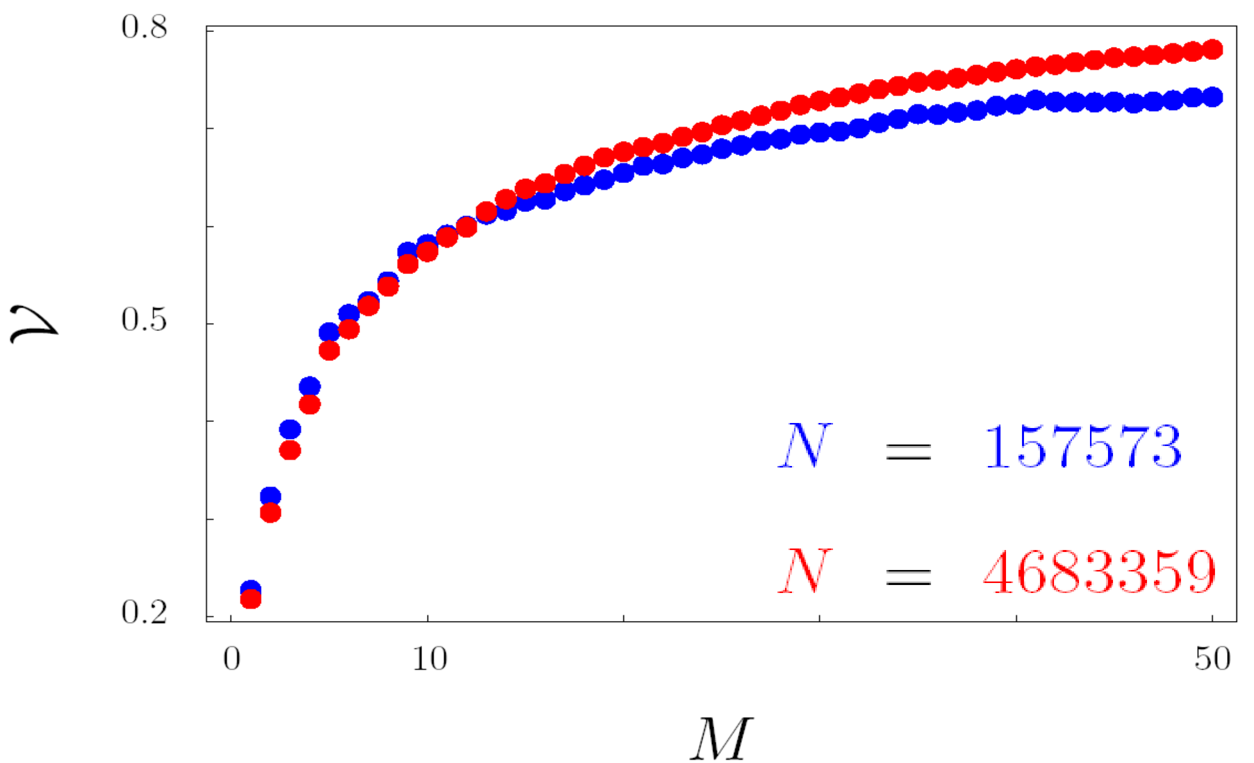
$$\mathcal{A}_N^{(M)}(\ell) = \frac{1}{M+1}\sum_{\ell=0}^{M} \exp\left(2\pi i\, m^2 \frac{N}{\ell}\right)$$

M=10

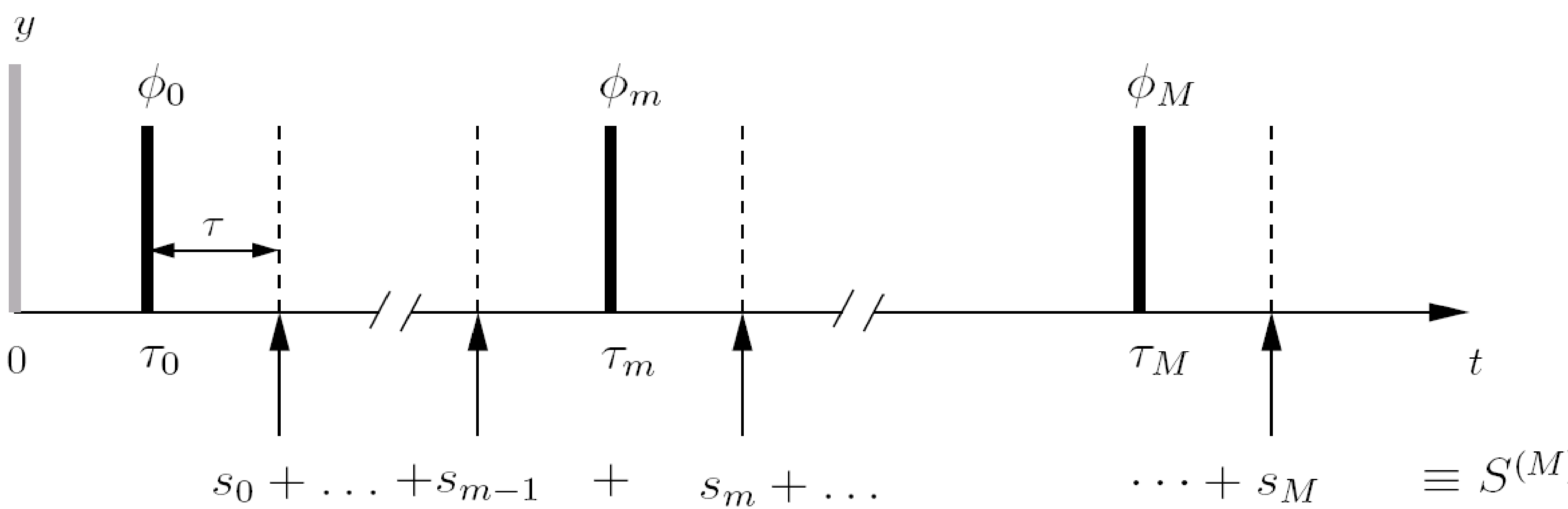


Contrast

$$\mathcal{V} = \frac{1-a}{1+a} \qquad a = \begin{pmatrix} \text{average of} \\ |\mathcal{A}_N^{(M)}(\ell)| \\ \text{for non-factors} \end{pmatrix}$$



NMR Implementation



$$H(t) = \hbar\omega_0\, I_z + 2\pi\hbar \sum_{k=0}^{M} \delta(t-\tau_k)\cos(\omega\tau_k - \phi_k)\, I_x$$

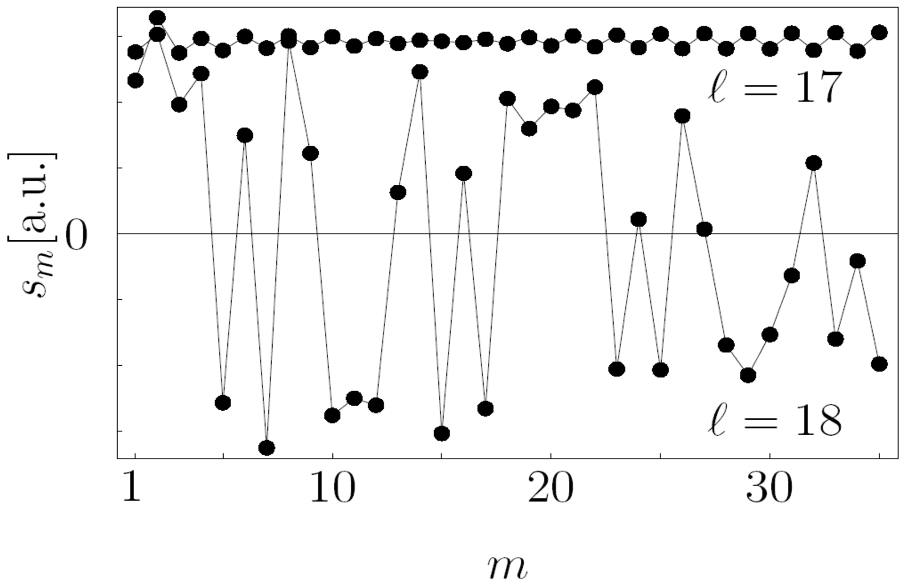
Echo

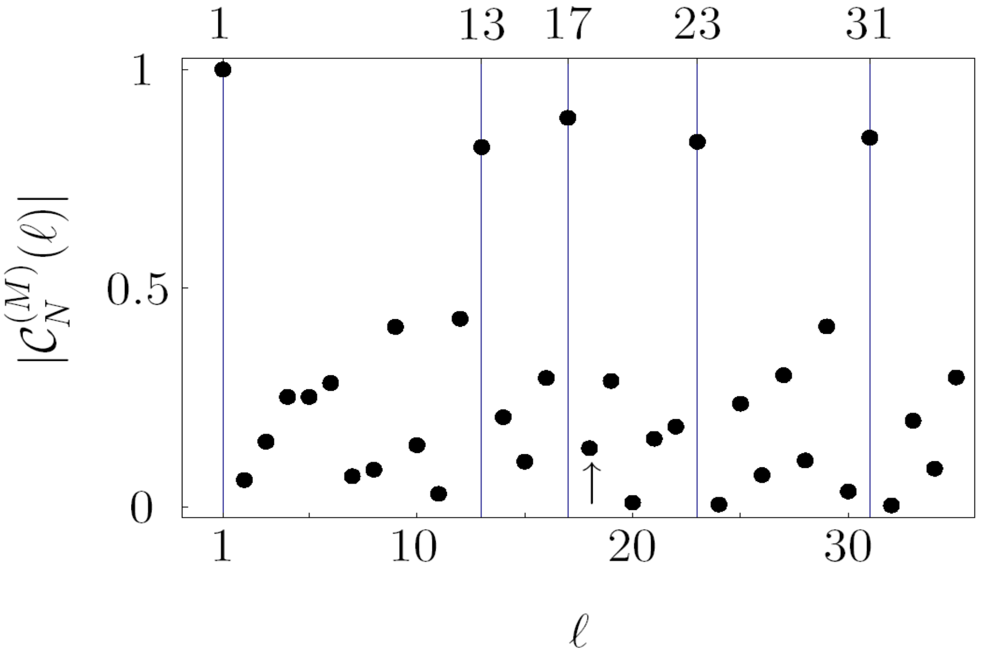
$$S^{(M)} = \sum_{m=0}^{M} \cos\left(\sum_{k=0}^{m} (-1)^k 2\phi_k\right)$$

$$\phi_k = \begin{cases} (-1)^k (2k-1)\pi \frac{N}{\ell} & \text{for } k \geq 1 \\ 0 & \text{for } k = 0 \end{cases}$$

$$\mathcal{C}_N^{(M)}(\ell) = \frac{1}{M+1} \mathcal{S}^{(M)} = \frac{1}{M+1} \sum_{m=0}^{M} \cos\left(2\pi\, m^2 \frac{N}{\ell}\right)$$

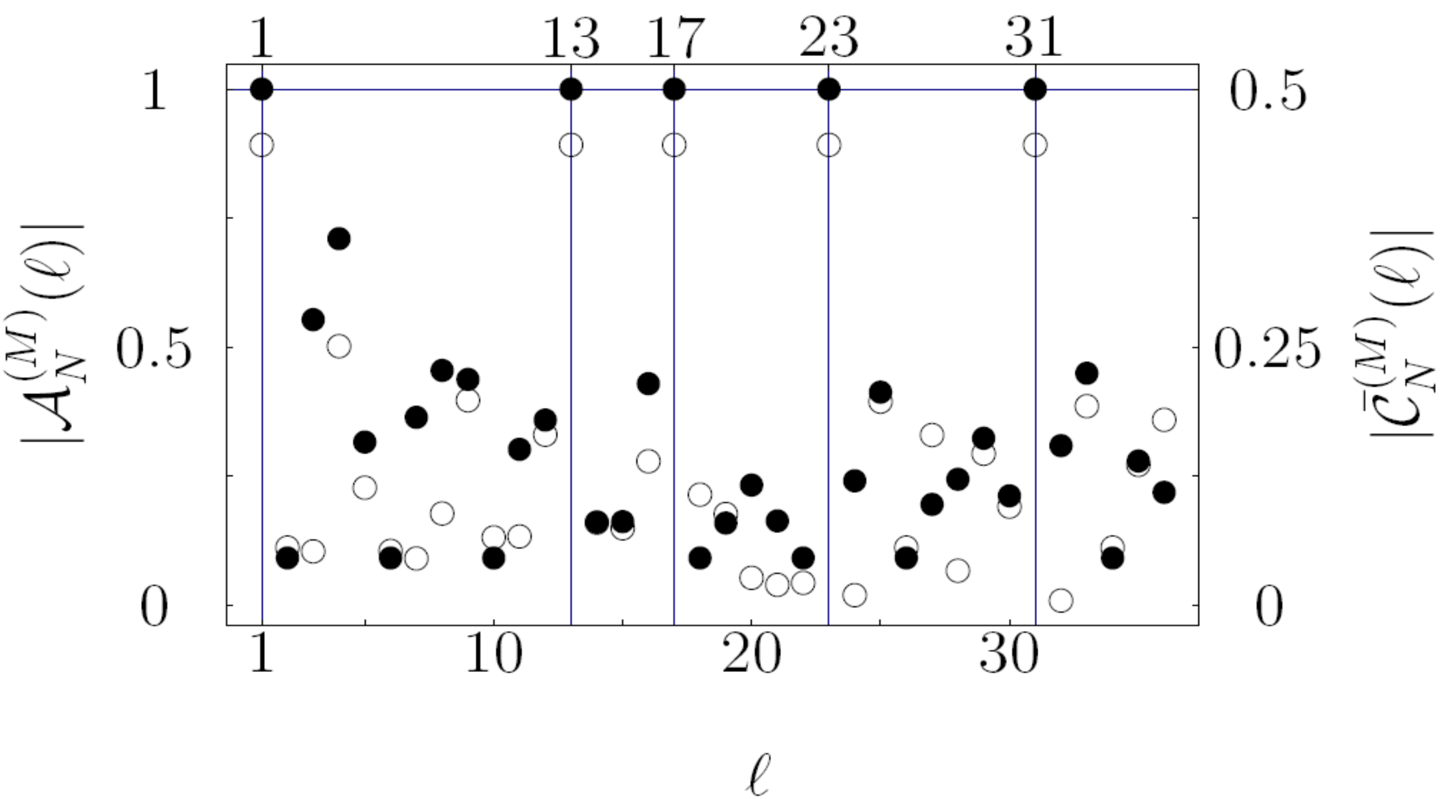
NMR Experiment: N=157573=13·17·23·31





Decoherence N=157573=13·17·23·31

$$\bar{\mathcal{C}}_N^{(M)}(\ell) = \frac{1}{M+1}\sum_{m=0}^{M} \exp\left(-m\frac{2\tau}{T_2}\right)\cos\left(2\pi\, m^2\frac{N}{\ell}\right)$$

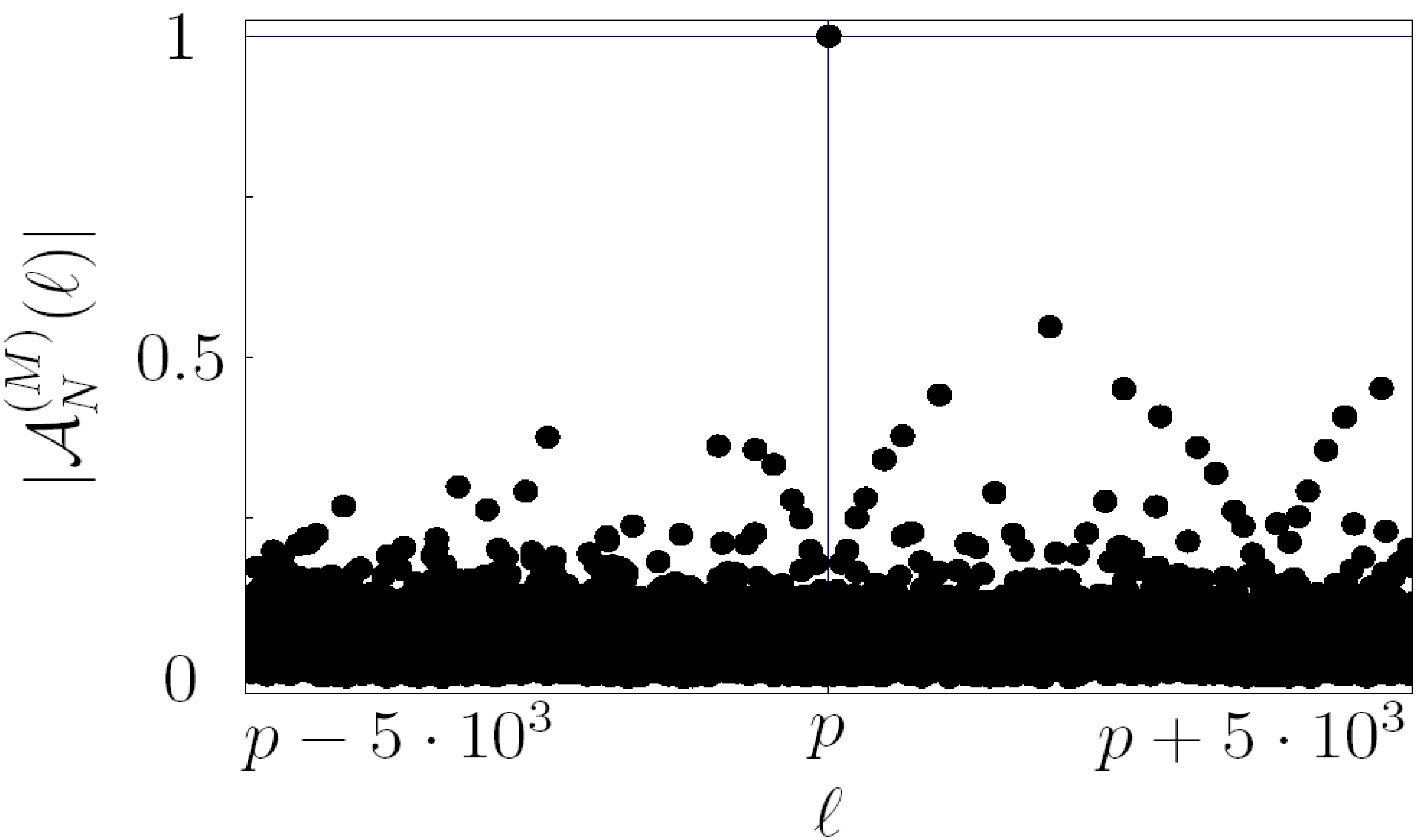


Probing the limits...

$$N = 1062885837863046188098307 = p \cdot q$$

$$p = 790645490053$$

M=200




GOVERNOR, IF WE WERE TO TURN OFF THE CAMERAS, WOULD YOU EXIST?
I DON'T WANT TO SPECULATE ON THAT.

J. P. Dahl	Lyngby
R. Mack	Ulm
H. Moya-Cessa	Puebla
W. Strunz	Freiburg
R. Walser	
W. P. Schleich	

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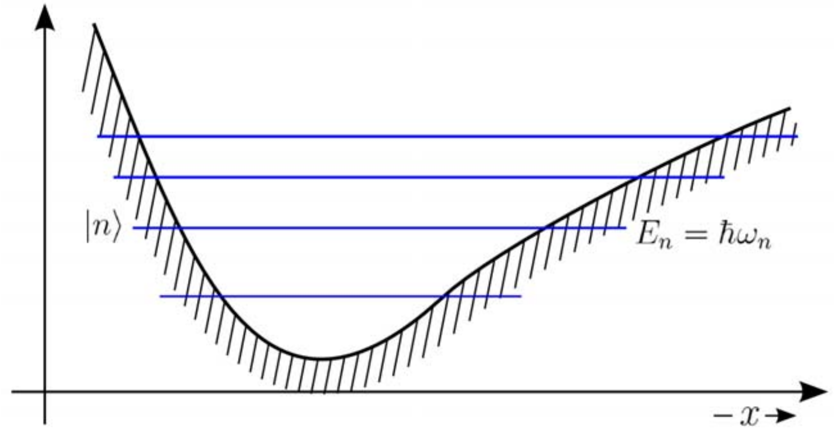
Autocorrelation function



$$|\psi(0)\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle$$

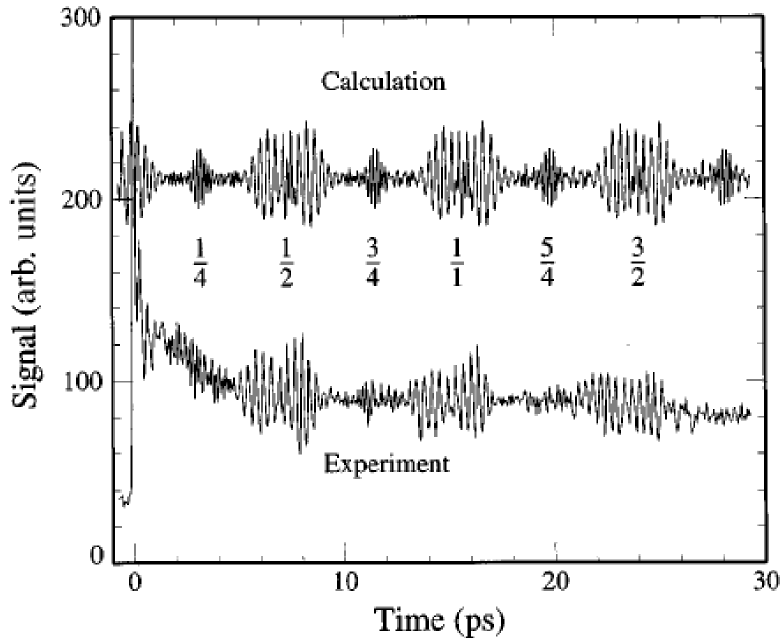
$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n e^{-i\omega_n t} |n\rangle$$

$$\mathcal{C}(t) \equiv \langle\psi(0)|\psi(t)\rangle = \sum_{n=0}^{\infty} |\psi_n|^2 e^{-i\omega_n t}$$

Observation of fractional revivals of a molecular wave packet

Marc J. J. Vrakking,* D. M. Villeneuve, and Albert Stolow
Steacie Institute for Molecular Sciences, National Research Council of Canada, Ottawa, Ontario, Canada K1A 0R6
(Received 27 October 1995)

The study and control of molecular dynamics through the use of femtosecond laser techniques require an understanding of various orders of molecular wave-packet phenomena such as revivals. We present the observation of fractional revivals in a vibrational wave packet: the *B* state of the Br_2 molecule. Fractional revivals occur in anharmonic systems at characteristic times in the nonclassical regime when, due to dephasing between vibrational eigenstates, the wave packet partially localizes into sub-wave-packets. Half and quarter revivals were clearly observed, as well as indications of one-sixth revivals. [S1050-2947(96)50707-9]



Observation of Rydberg Wave Packet Dynamics in a Coulombic and Magnetic Field

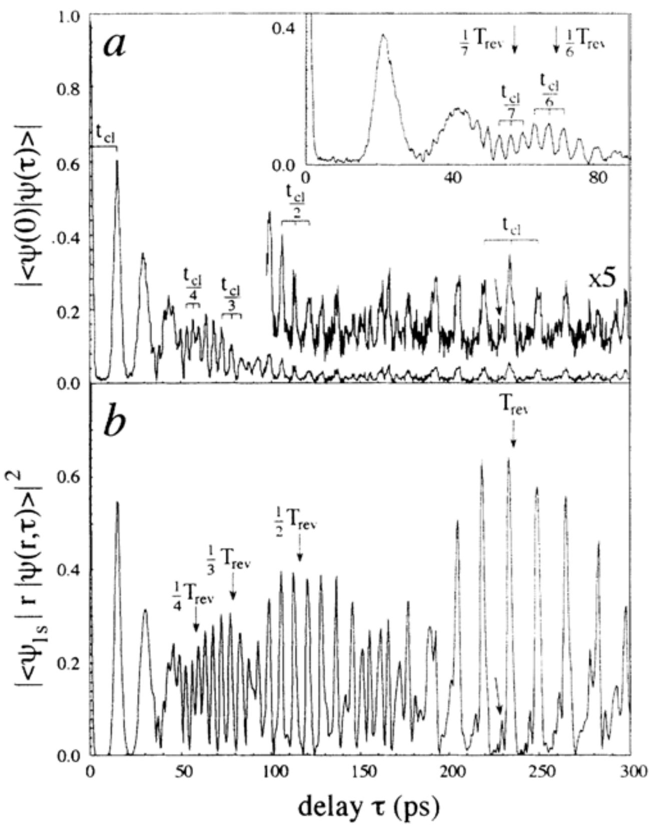
J. Wals,[1] H. H. Fielding,[1] J. F. Christian,[2] L. C. Snoek,[2] W. J. van der Zande,[2] and H. B. van Linden van den Heuvell[1,2]

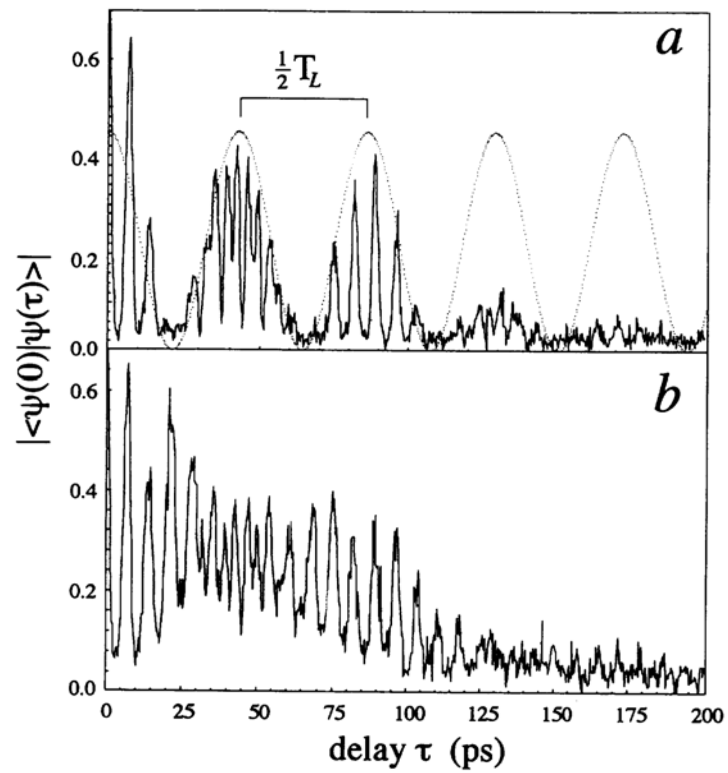
[1]*van der Waals-Zeeman Laboratory, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*

[2]*FOM-Institute for Atomic and Molecular Physics, Kruislaan 407, 1098 SJ, Amsterdam, The Netherlands*

(Received 17 January 1994)

Rydberg electron wave packets are investigated, using phase-modulated detection. In the absence of any external field we observe pure radial motion of a bound electron wave packet showing both the first integer revival and all fractional revivals up to the seventh order; in the presence of a magnetic field (0.81 T) both paramagnetic and diamagnetic effects are observed. Paramagnetism is visible in the time-correlation function of the wave packet as a sinusoidal intensity modulation, with a period of 43 ps, superimposed on the radial motion; the diamagnetism is observed as a decrease in the radial recurrence times (by up to 35%).





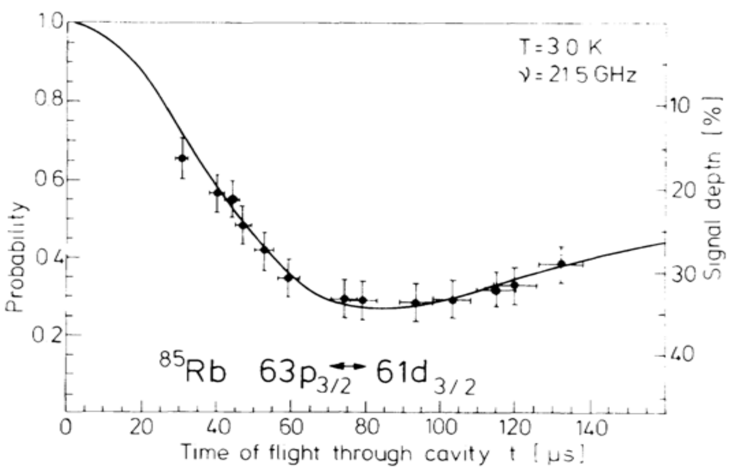
Observation of Quantum Collapse and Revival in a One-Atom Maser

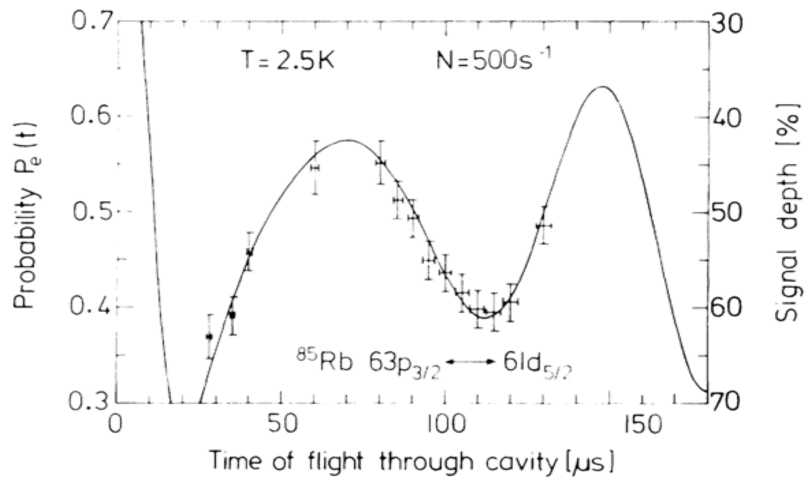
Gerhard Rempe and Herbert Walther
Sektion Physik, Universität München and Max-Planck-Institut für Quantenoptik, D 8046 Garching bei München, Federal Republic of Germany

and

Norbert Klein
Fachbereich Physik, Universität Wuppertal, D 5600 Wuppertal, Federal Republic of Germany
(Received 27 October 1986)

The dynamics of the interaction of a single Rydberg atom with a single mode of an electromagnetic field in a superconducting cavity was investigated. Velocity-selected atoms were used and the evolution of the atomic inversion as atom and field exchange energy was observed. The quantum collapse and revival predicted by the Jaynes-Cummings model were demonstrated experimentally for the first time. The evaluation of the dynamic behavior of the atoms allows us to determine the statistics of the few photons in the cavity.





$$P_e(t) \sim \sum_n W_n e^{-i\omega_n t} \qquad W_n \sim e^{-\sigma n}$$

$$\omega_n \sim \sqrt{n}$$

Special unharmonic oscillator

$$E_n = \hbar\omega \ln(n + a)$$

$$\psi_n = \mathcal{N} \frac{1}{(n + a)^\sigma}$$

$$\mathcal{C}_\sigma(t) = |\mathcal{N}|^2 \sum_{n=0}^{\infty} \frac{1}{(n + a)^\sigma} e^{-i\omega t \ln(n+a)}$$

$$\mathcal{C}_\sigma(\tau) = |\mathcal{N}|^2 \sum_{n=0}^{\infty} \frac{1}{(n + a)^{\sigma + i\tau}} \equiv \zeta(s, a)$$

$$s = \sigma + i\tau \equiv \sigma + i\omega t$$

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1 - \frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

Inverse problem

$$V(\xi) \quad \longrightarrow \quad E_n = ?$$

$$\underset{\text{Schrödinger equation}}{\uparrow}$$

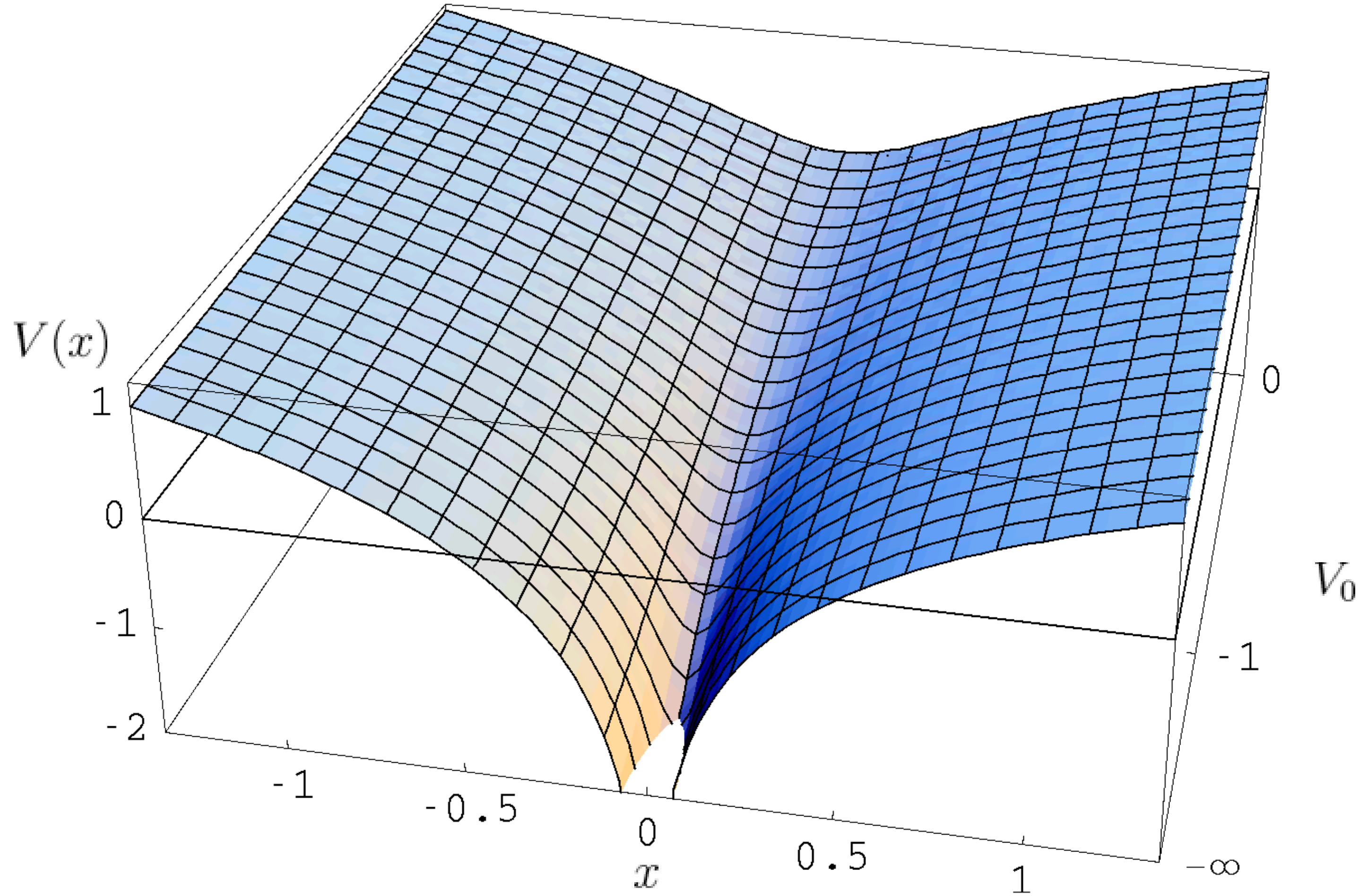
$$E_n = \hbar\omega \ln[\gamma(n+a)] \quad \longrightarrow \quad V(\xi) = ?$$

JWKB solution:

$$2\xi = e^{v^2(\xi)} \operatorname{erf}(v(\xi))$$

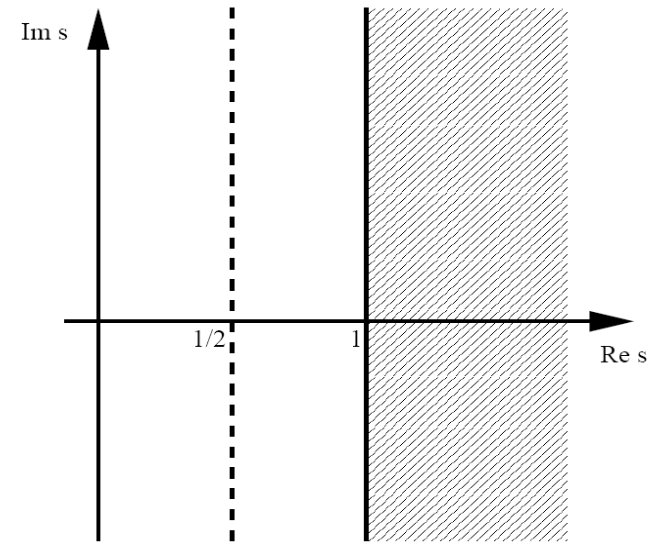
$$v(\xi) = \sqrt{\frac{V(\xi) - V_0}{\hbar\omega}}$$

Potential $E_n = \hbar\omega \ln(n + a)$



$$a = \begin{pmatrix} \text{Maslov} \\ \text{index} \end{pmatrix} + \frac{1}{\gamma} e^{V_0/\hbar\omega}$$

Riemann-Siegel formula



$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$s \equiv \sigma + it \qquad \mathrm{Re}\, s \equiv \sigma > 1$$

$$\zeta(s) \cong \sum_{n=1}^{N} \frac{1}{n^s} + \chi(s) \sum_{n=1}^{N} \frac{1}{n^{1-s}}$$

$$\chi(s) = \ldots$$

$$N \sim t^{1/2}$$

PHYSICAL REVIEW A VOLUME 44, NUMBER 3 1 AUGUST 1991

Nonclassical state from two pseudoclassical states

W. Schleich, M. Pernigo, and Fam Le Kien
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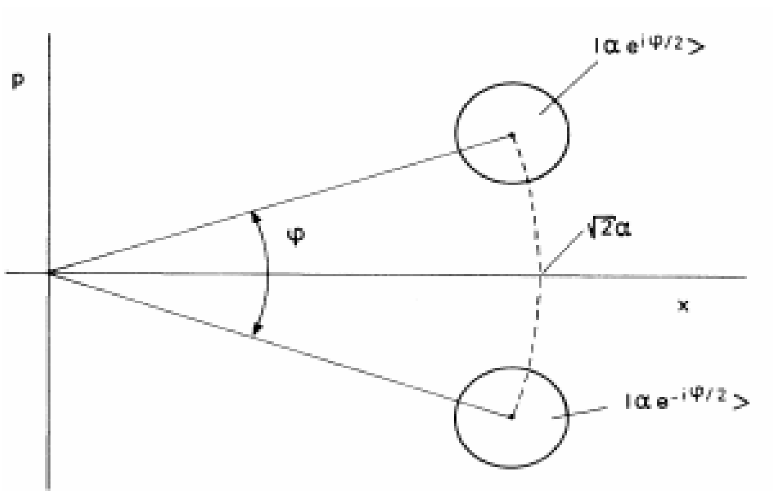
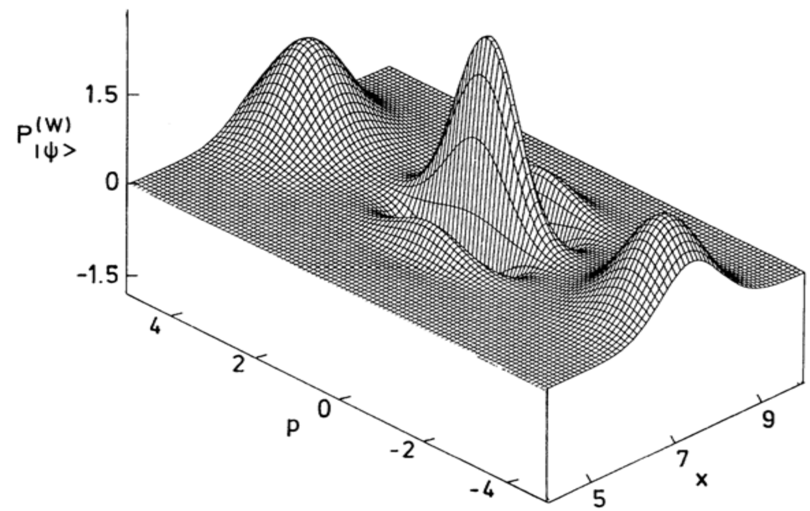
(Received 3 December 1990)

The quantum-mechanical superposition of two coherent states of identical mean photon number but different phases yields a state that can exhibit sub-Poissonian and oscillatory photon statistics, as well as squeezing.





$$|\psi\rangle \sim \sum_n w_n e^{in\varphi}|n\rangle + \sum_n w_n e^{-in\varphi}|n\rangle$$

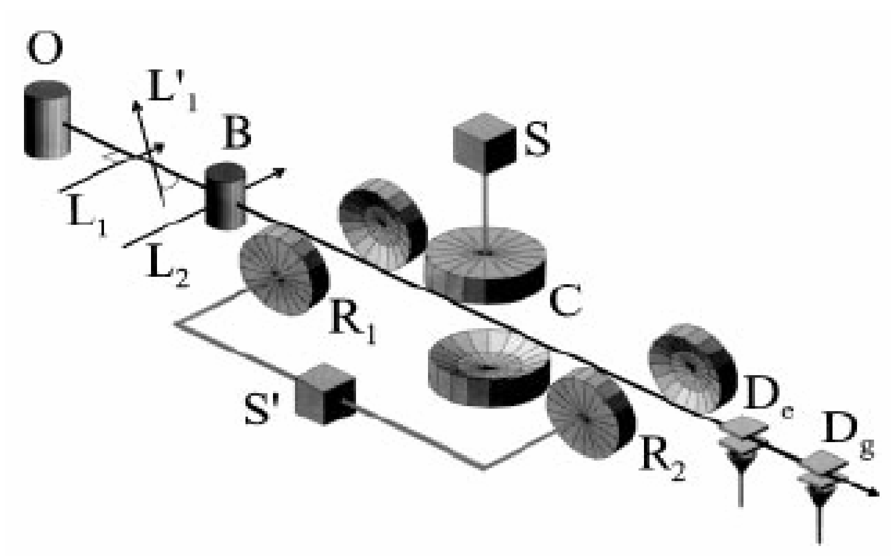
Observing the Progressive Decoherence of the "Meter" in a Quantum Measurement

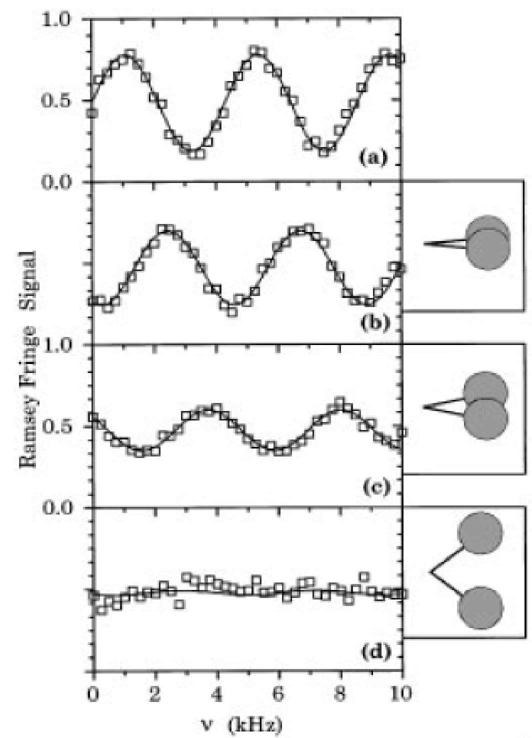
M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche

Laboratoire Kastler Brossel, Département de Physique de l'Ecole Normale Supérieure, 24 Rue Lhomond, F-75231 Paris Cedex 05, France*

(Received 10 September 1996)

A mesoscopic superposition of quantum states involving radiation fields with classically distinct phases was created and its progressive decoherence observed. The experiment involved Rydberg atoms interacting one at a time with a few photon coherent field trapped in a high Q microwave cavity. The mesoscopic superposition was the equivalent of an "atom + measuring apparatus" system in which the "meter" was pointing simultaneously towards two different directions—a "Schrödinger cat." The decoherence phenomenon transforming this superposition into a statistical mixture was observed while it unfolded, providing a direct insight into a process at the heart of quantum measurement. [S0031-9007(96)01848-0]





“Schrödinger cat”

Riemann zeta function:

$$\zeta = \sum_{n=1}^{N} \frac{1}{n^{\sigma}} e^{-i\varphi_n} + \chi(s) \sum_{n=1}^{N} \frac{1}{n^{1-\sigma}} e^{+i\varphi_n}$$

$$\varphi_n \equiv t \ln n$$

coherent state:

$$|\psi\rangle \sim \sum_n \psi_n e^{-i\phi_n} + \sum_n \psi_n e^{+i\phi_n}$$

$$\phi_n = \beta \cdot n$$

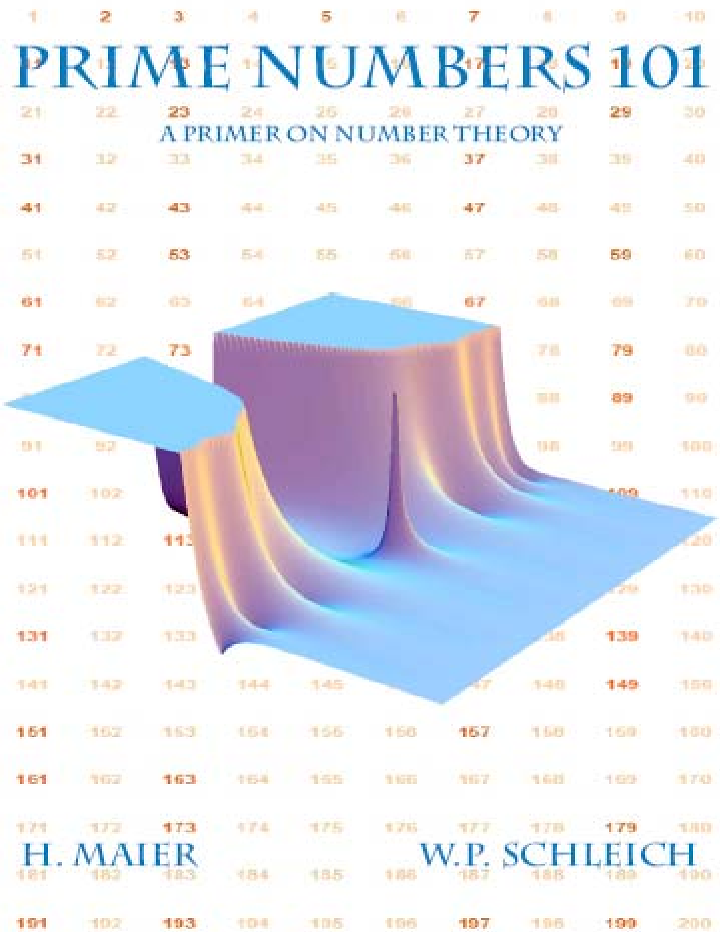
Analytic Continuation and Entanglement

coherent state:

$$H = \hbar\Omega\,\hat{n} + \frac{1}{2}\hbar\Omega\sigma_z + \hbar\omega\,\hat{n}\frac{1}{2}(\sigma_z + \mathbb{1})$$

Riemann zeta function:

$$H = \hbar\Omega\,\hat{n} + \frac{1}{2}\hbar\Omega\sigma_z + \hbar\omega\,\ln(n+1)\frac{1}{2}(\sigma_z + \mathbb{1})$$



- www.physik.uni-ulm.de/quan/book/Zahlenverschl.pdf
- password: numbers

Summary

- Factorization with Gauss sums
  - Scaling laws
  - Implementation in atoms
  - NMR experiment
- Wave packets and Riemann Zeta function
  - Autocorrelation function
  - Logarithmic spectrum
  - Schrödinger cats

Sidney Harris

"Actually I started out in quantum mechanics, but somewhere along the way I took a wrong turn."