

NON COMMUTATIVE GEOMETRY :

FUZZY SPACES, THE GROENEWOLD-MOYAL PLANE

THIS IS A LEAD TALK FOR WHAT
IS TO COME.

A TASTE OF THE FUTURE, ITS
INTIMATIONS WITH A PERSONAL FOCUS.

IT IS WELL-KNOWN THAT MULTANI
SOHAN HALWA DOES NOT EXIST OUTSIDE
LAHORE JUST AS THERE IS NO PIZZA
OUTSIDE NAPOLI.

MY TALK IS AT BEST A
FORETASTE OF SUCH DELICIOUS
FARE WHICH AWAITS YOU.

1
QUANTUM THEORY ON THE GROENEWOLD-
MOYAL PLANE, FUZZY SPACES

NONCOMMUTATIVE GEOMETRY IS A
BRANCH OF MATHEMATICS DUE TO
CONNES, RIEFFEL AND OTHERS.

PHYSICISTS IN A VERY SHORT TIME
VULGARISED IT AND USE THIS PHRASE
WHENEVER SPACETIME ALGEBRA IS
NONCOMMUTATIVE.

THERE ARE 2 SUCH PARTICULARLY
ACTIVE FIELDS
IN PHYSICS AT PRESENT:

• FUZZY PHYSICS.

• QFT (QUANTUM FIELD THEORY)
ON GROENEWOLD-MOYAL PLANE.

ITEM 1 IS EVOLVING INTO A TOOL
TO REGULATE QFT'S, AND FOR NUMERICAL WORK.

2

2 IT IS AN ALTERNATIVE TO LATTICE METHODS

ITEM 2 IS MORE A PROBE OF
PLANCK - SCALE PHYSICS.

THIS INTRODUCTORY TALK WILL FOCUS
ON ITEMS 1 & 2.

GROENEWOLD-⁴³
MOYAL PLANE HAS ALSO EMERGED
FROM STRING PHYSICS, AS ALSO FUZZY SPACES.
EACH BEING ALSO A SIMPLE NONCOMMUTATIVE MODEL, MERITS STUDY.

HISTORY

GROENEWOLD-
MOYAL PLANE :

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}$$

GROEN-MOYAL PLANE FIRST OCCURS IN A
LETTER FROM HEISENBERG TO PEIERLS.
HE SUGGESTS THAT THE UNCERTAINTY
PRINCIPLE $\Delta x_\mu \Delta x_\nu \gtrsim \frac{1}{2} |\theta_{\mu\nu}|$ WILL
PROVIDE SHORT DISTANCE CUT-OFF &
REGULATE QFT'S. HE COMPLAINS^{ABOUT} HIS
IGNORANCE OF MATHEMATICS TO STUDY
THIS POSSIBILITY.

PEIERLS COMMUNICATES THIS IDEA
TO PAULI, AND PAULI TO SNYDER.

SNYDER WRITES FIRST PAPER ON
THE SUBJECT. THIS FOLLOWED BY A
PAPER OF YANG. JOE WEINBERG, SYRACUSE,
THE "OTHER" MAN IN OPPEINHEIMER CASE SV ARCHIVES

IN MID-90'S, DOPLICHER, FREDENHAGEN
AND ROBERTS SYSTEMATICALLY CONSTRUCTED
UNITARY QFT'S ON ^{GROENEWOLD-}MOYAL PLANE, EVEN
WITH TIME-SPACE NONCOMMUTATIVITY.

LATER STRING PHYSICS ENCOUNTERED
THESE STRUCTURES

WHAT IS NONCOMMUTATIVE GEOMETRY

ACCORDING TO CONNES, NONCOMMUTATIVE
GEOMETRY IS A SPECTRAL TRIPLE, "HOLY TRINITY",
($A, D, \mathcal{H}$)

A = A C^* -ALGEBRA, POSSIBLY NONCOMMUTATIVE

D = DIRAC OPERATOR.

$\mathcal{H}$ = A HILBERT SPACE ON WHICH THEY
ARE REPRESENTED.

IF A IS COMMUTATIVE, WE CAN
RECOVER A HAUSDORFF TOPOLOGICAL
SPACE ON WHICH A ARE FUNCTIONS,
ITS DIFFERENTIABLE STRUCTURE, ...

(GEL'FAND - NAIMARK, CONNES, ...)

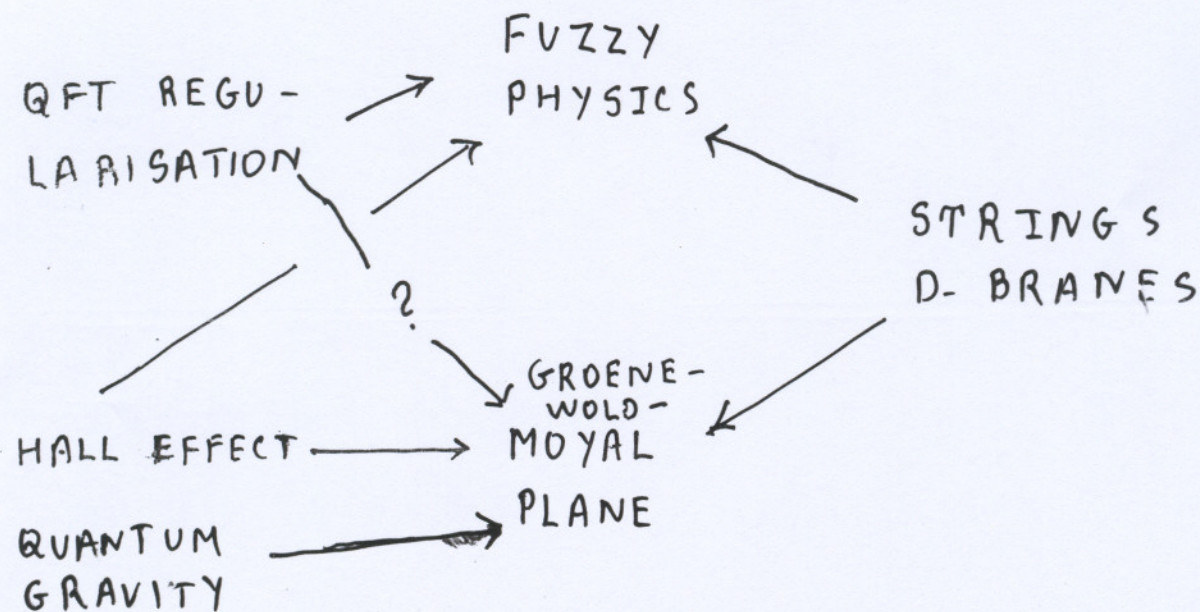
(THERE ARE ADDITIONAL AXIOMS, DUE TO CONNES)

A CLASS OF EXAMPLES WITH A NONCOM-
MUTATIVE IS DUE TO CONNES AND LANDI.

MANY EXAMPLES IF SOME AXIOMS NOT
ENFORCED: $SU(2)_q$, FUZZY SPACES, GROENE-
MOYAL PLANE, ...

CONCEPTUAL REVOLUTION : MANIFOLDS
ARE BEING REPLACED BY THEIR "DUALS,"
ALGEBRAS, MUCH AS IN QUANTUM
MECHANICS.

THE TANGLED WEB



WHAT IS FUZZY PHYSICS?

2 DIMENSIONAL EXAMPLE : S^2 WE QUANTISE S^2 TO REGULARISE,INTRODUCE SHORT DISTANCE CUT-OFF[†]

$$\text{THUS: } S^2 : [\vec{x} \in \mathbb{R}^3 : \vec{x} \cdot \vec{x} = R^2]$$

NOW CONSIDER ANGULAR MOMENTA

 L_i :

$$[L_i, L_j] = i \epsilon_{ij} L_k ; \quad \vec{L}^2 = \ell(\ell+1).$$

SET

$$\hat{x}_i = \frac{R L_i}{[\ell(\ell+1)]^{1/2}} \Rightarrow$$

$$\hat{x} \cdot \hat{x} = R^2, \quad [\hat{x}_i, \hat{x}_j] = \frac{R}{[\ell(\ell+1)]^{1/2}} i \epsilon_{ij} R \hat{x}_k.$$

 $\hat{x}_i \in \text{Mat}_{2\ell+1}$. AS $\ell \rightarrow \infty$, THEY

BECOME COMMUTATIVE. THEY GIVE
FUZZY SPHERE S_F^2 , RADIUS R ,
OF DIMENSION $2\ell+1$.

G'

+

THUS EX. IN CLASSICAL MECHANICS, # OF STATES IN PHASE SPACE VOLUME

$$\Delta v : d^3p d^3q$$

IS ∞ .

BUT WE KNOW SINCE PLANCK AND BOSE THAT ON QUANTISATION, IT BECOMES

$$\Delta v / h^3 : \text{FINITE.}$$

THIS IS THE IDEA.

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WHY IS THIS SPACE FUZZY?

AS $\hat{x}_i, \hat{x}_j$ ($i \neq j$) DO NOT COMMUTE

WE CANNOT SHARPLY LOCALISE $\hat{x}_i$.

ROUGHLY IN VOLUME ^(AREA) $4\pi R^2$, THERE

ARE $(2\ell+1)$ STATES OR $\approx$ STATES

PER UNIT VOLUME = $(2\ell+1)/(4\pi R^2)$.

SCALAR FIELD Φ = POLYNOMIAL IN $\hat{X}_L$
 = $(2\ell+1)$ - DIMENSIONAL MATRICES.

DIFFERENTIATION: $\mathcal{L}_L \Phi = [L_L, \Phi]$:
 INFINITESIMAL ROTATION.

A SCALAR FIELD ACTION:

$$S = \lambda_1 \text{Tr} [\mathcal{L}_L \Phi]^\dagger [\mathcal{L}_L \Phi] + \lambda_2 \text{Tr} (\Phi^\dagger \Phi)^2$$

SIMULATIONS: X. MARTIN, MEDINA
 ET. AL.

MAJOR FINDINGS:

CONTINUUM LIMIT EXISTS (2ND
 ORDER PHASE TRANSITION).

IF $\Phi = \sum c_{\ell m} \hat{Y}_{\ell m}$, $\hat{Y}_{\ell m}$ = SPHERI-
 CAL TENSORS, 3 PHASES:

DISORDERED: $\langle \sum |c_{\ell m}|^2 \rangle = 0$

UNIFORM ORDERED: $\langle |c_{00}|^2 \rangle \neq 0$, $\langle |c_{\ell m}|^2 \rangle$
 $\ell \neq 0$.

NON-UNIFORM ORDERED: ALL BUT
 $\langle |e_{im}|^2 \rangle$ ZERO, $\langle |c_{im}|^2 \rangle \neq 0$.

LAST: ANALOGUE OF GUPSER-SONDHI
 STRIPED PHASE.

— S_F^2 HAS DIRAC OPERATOR WITH
 NO FERMION DOUBLING, INSTANTONS,

... SEEMS BETTER THAN LATTICE FOR SYMMETRIES

SUSY: REPLACE $SU(2)$ BY $OSP(2,1) \Rightarrow$

FUZZY SPHERE BECOMES FUZZY $N=1$ SUS

SUPERS PHERE. SIMULATIONS STARTING.

— STRINGS: MYERS: IF N D-BRANES
 ARE CLOSE, TRANSVERSE COORDI-
 NATES Φ_i BECOME $N \times N$
 MATRICES. ACTION:

$$S = \lambda \text{Tr} [\Phi_i, \Phi_j]^\dagger [\Phi_i, \Phi_j] + i f_{ijk} \text{Tr} \Phi_i \Phi_j \Phi_k$$

f_{ijk} TOTALLY ANTISYMMETRIC.

$\delta S = 0 \Rightarrow [\Phi_i, \Phi_j] \approx i f_{ijk} \Phi_k$ GIVE
SOLUTIONS IF f_{ijk} STRUCTURE
CONSTANTS OF SIMPLE COMPACT
LIE GROUP.

THUS POSSIBILITIES :

$$\Phi_i = e L_i, \quad f_{ijk} = e \epsilon_{ijk}, \quad e = \text{CONS.}$$

TANT. WE CAN HAVE

$$L_i \text{ IRREDUCIBLE, } \vec{L} \cdot \vec{L} = \ell(\ell+1), 2\ell+1=N$$

WRITE $L_i = L_i^{(\ell)}$. ONE FUZZY SPHERE.

$$L_i = \bigoplus_K L_i^{(\ell_K)} : \text{MANY FUZZY SPHERES.}$$

$$\sum (2\ell_K + 1) = N.$$

II

STABILITY ANALYSIS OF THESE
SOLUTIONS AND STUDIES OF THEIR
PHASES, INCLUDING NUMERICALLY,
BY MANY GROUPS. (TIFR GROUP)

GROENEWOLD -
MOYAL PLANE

(IT IS AN ALTERNATIVE TO LATTICE METHODS.)

ITEM 2 IS MORE A PROBE OF
 PLANCK-SCALE PHYSICS.

THIS INTRODUCTORY TALK WILL
 FOCUS ON ITEMS 1, 2 AS ^{THEY} ~~IT~~ WILL
 DOMINATE THIS MEETING.

QUANTUM GRAVITY & SPACETIME NONCOMMUTATIVITY : HEURISTICS.

(DOPLICHER, FREDENHAGEN, ROBERTS)

1) SPACE-SPACE NONCOMMUTATIVITY:

COMPTON WAVELENGTH $\hbar/mc$ TO

PROBE PLANCK SCALE L REQUIRES

$\hbar/mc \lesssim L$ OR $m \gtrsim \hbar/Lc \gtrsim \text{PLANCK}$

MASS. SUCH HIGH MASS IN THIS SMALL

VOLUME $\sim L^3$ WILL STRONGLY CHANGE GRAVITY.

CAN BECOME BLACK HOLE. SUGGESTS A
FUNDAMENTAL LENGTH LIMITING
SPATIAL LOCALISATION.

2) SIMILAR ARGUMENTS CAN BE MADE
ABOUT TIME-LOCALISATION $\Rightarrow$ OBSERVA-
TION OF VERY SHORT TIME SCALES
REQUIRE VERY HIGH ENERGIES. THEY
CAN PRODUCE BLACK HOLES. BLACK
HOLE HORIZON WILL THEN LIMIT
SPATIAL RESOLUTION SUGGESTING
 $\Delta t \Delta |\vec{x}| \geq L^2$, L : A FUNDAMENTAL
LENGTH.

GROENEWOLD -
THE $\star$ MOYAL PLANE

$x_\mu \star x_\nu - x_\nu \star x_\mu = -i\theta_{\mu\nu}$, x_μ : COORDINATE
FUNCTIONS, MODELS ABOVE SPACETIME
UNCERTAINTIES.

IN MID-90'S, DOPLICHER, FREDEN-
 HAGEN AND ROBERTS SYSTEMATICALLY
 DEVELOPED UNITARY QFT'S ON
 GROENEWOLD-
 MOYAL PLANE EVEN WITH TIME-
 SPACE NONCOMMUTATIVITY.

EMERGES ALSO FROM STRING
 PHYSICS & QUANTUM HALL EFFECT.

WHAT IS ^{GROENEWOLD-}
MOYAL PLANE $\mathcal{A}_\theta(\mathbb{R}^{d+1})$

CONSISTS OF FUNCTIONS $\alpha, \beta, \dots$ ON
 $\mathbb{R}^{d+1}$ WITH $\ast$ -PRODUCT

$$\alpha \ast \beta = \alpha e^{-i/2 \overleftarrow{\partial} \wedge \overrightarrow{\partial}} \beta, \quad \overleftarrow{\partial} \wedge \overrightarrow{\partial} = \overleftarrow{\partial}_\mu \Theta^{\mu\nu} \overrightarrow{\partial}_\nu$$

IMPLIES FOR SPACETIME COORDINATES:

$$[x_\mu, x_\nu]_\ast = -i \Theta_{\mu\nu}.$$

$$\hookrightarrow x_\mu \ast x_\nu - x_\nu \ast x_\mu.$$

QHE, HOW THIS EMERGES FROM STRING PHYSICS

WE SAW :

• HEISENBERG'S ARGUMENTS FOR MOYAL PLANE FOR REGULATING QFT.

• DOPLICHER ET AL. ARGUMENTS FROM QUANTUM GRAVITY.

A 3RD SOURCE IS QUANTUM GRAVITY HALL EFFECT, AND A SIMILAR EFFECT IN STRING THEORY. THUS:

QHE (LANDAU PROBLEM).

ELECTRON IN 1-2 PLANE, $\vec{B} = (0, 0, B)$

$$\Rightarrow A_a = -\frac{B}{2} \epsilon_{ab} x^b \quad a, b = 1, 2.$$

$$\Rightarrow L = \frac{m}{2} \dot{x}_a^2 - e \dot{x}_a A_a$$

IF $eB \rightarrow \infty$,

$$L \approx + \frac{eB}{2} (\dot{x}_1 x_2 - \dot{x}_2 x_1)$$

$$\Rightarrow [\hat{x}_a, \hat{x}_b] = + \frac{i}{eB} \epsilon_{ab} \text{ ON QUANTISATION}$$

THIS RESULT, FOR $B \rightarrow \infty$, IS EXACT.
 THEN ONLY LOWEST LANDAU LEVEL
 POPULATED. IF P PROJECTOR TO
 THIS LEVEL AND $\hat{X}_a$ ACTUAL POSITION
 OPERATOR,

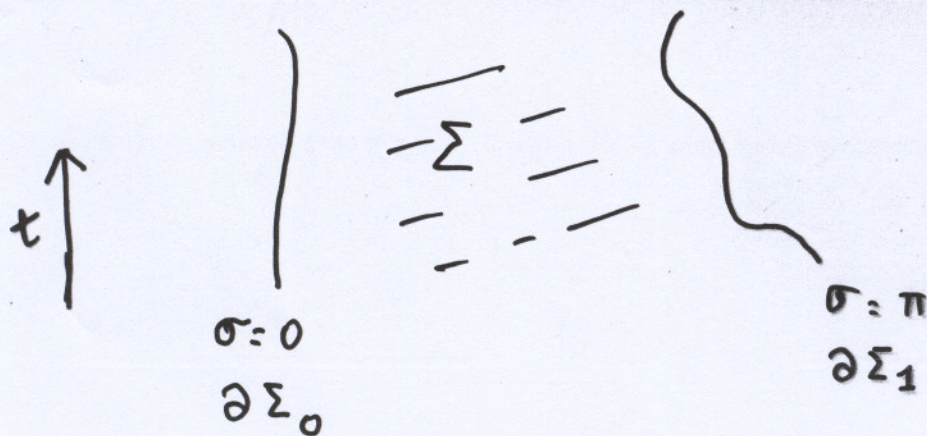
$$\hat{x}_a = P \hat{X}_a P.$$

STRINGS

CONSIDER OPEN ~~BO~~ STRINGS ENDING
 ON Dp -BRANES.

IF $B_{ij} = -B_{ji} = \text{CONSTANT}$ 2-FORM
 (NEVEU-SCHWARZ) FIELD

$$S_{\Sigma} = \frac{1}{4\pi\alpha'} \int_{\Sigma} [g_{ij} \partial_a x^i \partial_a x^j - 2\pi\alpha' B_{ij} \partial_a x^i \times \partial_b x^j \epsilon^{ab} + \text{SPINOR TERMS}] d\sigma dt$$



$$\text{As } d' \rightarrow \infty, \quad S_{\Sigma} = -\frac{2\pi e}{4\pi} \int_{\Sigma} B_{ij} dx^i \wedge dx^j$$

$$= -\frac{1}{2} \left[\int_{\partial\Sigma_0} - \int_{\partial\Sigma_1} \right] e B_{ij} x^i \frac{dx^j}{dt}$$

$$\Rightarrow e [B_{ij} \hat{x}^j, \hat{x}_k] = i \delta_{ik} \quad \text{OR}$$

$$[\hat{x}_j, \hat{x}_k] = \frac{i}{e} (B^{-1})_{jk} : \text{THE "G}^M\text{" PLANE.}$$

PREHISTORY (BEFORE $\sim 2004/05$)

MUCH WORK ON

Ⓐ • QFT'S AND RENORMALISATION THEORY.

Ⓑ • PHENOMENOLOGY: LORENTZ - GROUP VIOLATION, CP, T, CPT, ...

↳ FROM $\theta_{\mu\nu}$ IN $[x_\mu, x_\nu]_* = i\theta_{\mu\nu}$
BUT IN $\sim 04, 05$, CHAICHIAN ET AL, ASCHIERI ET AL POPULARISE

DRIN'FELD TWIST WHICH RESTORES
FULL $\overset{E}{\text{DIFFOMORPHISM}}$ INVARIANCE λ
(WITH A TWIST!)
DESPITE THE PRESENCE OF CONSTAN

$\theta_{\mu\nu}$ IN $[x_\mu, x_\nu]_* = i\theta_{\mu\nu}$.

THIS TWIST ALSO $\overset{S}{\text{TWISTS}}$ STATISTICS (BAL ET AL.).

MUCH OF THIS KNOWN FOR SOME TIME TO MATID, OECKL, FIORE,

SO DRIN'FELD TWIST TWISTS BOTH

1) ACTION OF DIFFEOS AND

2) STATISTICS.

THIS BRINGS INTO QUESTION MUCH OF PREHISTORY-ANALYSIS.

EXAMPLE : 1) LORENTZ INVARIANCE
NEED NOT BE VIOLATED EVEN IF
 $\Theta_{\mu\nu} \neq 0$. 2) THERE NEED BE NO
ULTRAVIOLET- INFRARED (UV-IR) MIXING
(SEE LATER). IN ABSENCE OF GAUGE FIELDS.

THERE IS ALSO A STRIKING, CLEAN
SEPARATION OF MATTER FROM
GAUGE FIELDS DUE TO DRIN'FELD
TWIST. [†] REMINISCENT OF SEPARATION
OF PARTICLES AND WAVES IN CLASSI-
CAL THEORY. [†] THEY HAVE TO BE
TREATED DIFFERENTLY (BALET_{et al})

THERE IS MUCH MORE TO
TALK ABOUT : STANDARD MODEL,
GRAVITY, PHENOMENOLOGY, ...
MUCH ROOM FOR FANTASY.

EXERCISE IMAGINATION

NOT ALL THE NAMES

SECKIN KURKCUOGLU - ANKARA

BABAR QURESHI - LAHORE

SACKIN VAIDYA	}	INDIA
KUMAR GUPTA		
T. R. GOVINDARAJAN		

A. PINZUL : MOSCOW

GIANPIERO MANGANO : NAPOLI.