Strings, Black Holes and Modular Forms

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Earlier work:

R. Dijkgraaf, E. Verlinde, H. Verlinde, hep-th/9607026

+ many other follow up papers

String theory is an attempt to construct a unified theory of matter and forces acting between matter.

It is based on the hypothesis that the basic building blocks of matter are not particles but tiny elementary strings.

The size of the string is smaller than the resolution of the most powerful microscope available to us today.

Hence a generic vibrational state of the string looks like a particle.

String theory can be formulated only if the total dimension of space is 9 and not 3.

Nevertheless we can recover our 3 dimensional world from string theory if the 9 dimensional space is a product

$$R^3 \times \mathcal{M}$$

R^3: 3 dimensional Euclidean space

$\mathcal{M}$: a 6 dimensional compact space of small size

e.g. product of six small circles.

Different choices of $\mathcal{M}$ give rise to different phases of string theory.

Our world would correspond to some particular choice of $\mathcal{M}$.

Nevertheless we need to explore the physics of different possible $\mathcal{M}$ in order to unravel the complete structure of string theory.

One of the important questions to study in any of these phases is the charge spectrum.

How many different stable particles are there with a given amount of charge?

My talk today will be based on the study of type IIA string theory for a specific class of $\mathcal{M}$ known as CHL compactification.

$\mathcal{M}$: labelled by a prime number $N = 1, 2, 3, 5, 7$

Construction of $\mathcal{M}$

1. First consider a compact space $K3 \times S^1 \times \widetilde{S}^1$

$K3$: A well known four dimensional manifold

$S^1, \widetilde{S}^1$: two circles labeled by $y, \widetilde{y}$ of period 2π

Points on $K3 \times S^1 \times \widetilde{S}^1$: $(P, y, \widetilde{y})$, $\quad P \in K3$

2. Now identify the points

$$(P, y, \widetilde{y}) \text{ and } (P', y + \tfrac{2\pi}{N}, \widetilde{y})$$

P' and P are related by an order N discrete symmetry transformation of $K3$.

We shall study charge spectrum in these phases.

These phases of string theory typically have more than one electromagnetic field.

Thus a particle is labelled by not one charge but a set of charges

$$\vec{Q} = (Q_1, Q_2, \ldots Q_r)$$

$$r = \frac{48}{N+1} + 4$$

A generic particle in this theory also carries magnetic charge.

Thus we also need a magnetic charge vector to label the particle

$$\vec{M} = (M_1, M_2, \ldots M_r)$$

Question: What is the number of stable particles carrying electric charge $\vec{Q}$ and magnetic charge $\vec{M}$?

$\rightarrow$ should be a function $d(\vec{Q}, \vec{M})$

In string theory exact answers are very hard to obtain.

Most of the results in string theory are obtained as a perturbation expansion in some small parameter labelling the compactification.

In few cases one can ‘guess’ exact answers for some quantities and then test if it satisfies various consistency requirements.

In even fewer cases one can actually compute exact answers.

Our answer for $d(\vec{Q}, \vec{M})$ involves

guesswork $+$ consistency tests

A better understanding of the answer may eventually lead to a proof.

What are the consistency tests?

1. $d(\vec{Q},\vec{M})$ must be integer for every $(\vec{Q},\vec{M})$.

2. $d(\vec{Q},\vec{M})$ must be consistent with the symmetries of the theory.

If the theory has a symmetry that relates a state of charge $(\vec{Q},\vec{M})$ to another state of charge $(\vec{Q}',\vec{M}')$, then

$$d(\vec{Q},\vec{M}) = d(\vec{Q}',\vec{M}')$$

3. For large values of charges the particle acquires a large mass.

$\longrightarrow$ produces strong gravitational attraction.

For large enough charges the particle is so heavy that even light cannot escape its gravitational pull.

$\longrightarrow$ it behaves as a 'black hole'.

In quantum theory a black hole behaves as a thermodynamic object with definite entropy $S(\vec{Q}, \vec{M})$.

This entropy can be calculated from purely geometric data.

On the other hand in statistical mechanics

$$S(\vec{Q}, \vec{M}) = \ln d(\vec{Q}, \vec{M})$$

Thus knowledge of entropy provides information about how $d(\vec{Q}, \vec{M})$ should behave for large charges.

Question: Can we find a $d(\vec{Q}, \vec{M})$ that satisfies all the three requirements?

Use of symmetry

CHL compactifications have some symmetries which act on $\vec{Q}$ and $\vec{M}$ separately:

$$\vec{Q}' = U\,\vec{Q}, \qquad \vec{M}' = U\,\vec{M}$$

U is an $r \times r$ matrix with integer entries satisfying

$$U^T L U = L$$

L is a fixed matrix with 6 eigenvalues $+1$ and $(r-6)$ eigenvalues -1.

$$\vec{Q}' = U\,\vec{Q}, \quad \vec{M}' = U\,\vec{M}, \quad U^T L U = L$$

We can make $d(\vec{Q}, \vec{M})$ invariant under this symmetry by taking it as a function of invariant bilinear forms:

$$d(\vec{Q}, \vec{M}) = f(\vec{Q}^2, \vec{M}^2, \vec{Q} \cdot \vec{M})$$

where

$$\vec{Q}^2 = Q^T L Q, \quad \vec{M}^2 = M^T L M, \quad \vec{Q} \cdot \vec{M} = Q^T L M$$

This automatically satisfies $d(\vec{Q}, \vec{M}) = d(\vec{Q}', \vec{M}')$

S-duality symmetry:

$$d(\vec{Q}, \vec{M}) = d(\vec{Q}', \vec{M}')$$

for

$$\begin{pmatrix} \vec{Q}' \\ \vec{M}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{M} \end{pmatrix}$$

where a, b, c, d are intergers satisfying

$$ad - bc = 1, \quad (a-1),\ c \text{ are multiples of } N$$

This gives non-trivial condition on $f(k, l, m)$.

Thus we are looking for a function $f(k,l,m)$ of three variables which satisfy the conditions:

1. $f(k,l,m)$ is integer for every allowed value of (k,l,m).

2. It must not change under an S-duality transformation.

3. For large values of k,l,m it should agree with the result of black hole entropy calculation.

Note: Electric and magnetic charges are quantized $\rightarrow$ k,l,m take discrete values.

Generating function

Introduce a set of new variables t, u, v and define

$$g(t,u,v) = \sum_{k,l,m} f(k,l,m) e^{2\pi i(k\,t + l\,u + m\,v)}$$

Knowing $g(t,u,v) \leftrightarrow$ knowing $f(k,l,m)$

The symmetry requirement + other requirements on f

$\rightarrow$ specific requirements on $g(t,u,v)$.

Question: Can we guess a form of $g(t,u,v)$ satisfying these requirements?

One can construct a function satisfying these requirements.

Furthermore this function has some nice mathematical properties.

$\rightarrow$ gives hope that it may be possible to eventually prove this conjecture.

The answer:

$$g(t,u,v) = \frac{1}{\Phi(t,u,v)}$$

$\Phi(t,u,v)$: A Siegel modular form of a subgroup of $Sp(2,Z)$

Define

$$\Omega = \begin{pmatrix} t & v \\ v & u \end{pmatrix}, \qquad k + 2 = 24/(N+1)$$

Then

$$\Phi((A\Omega{+}B)(C\Omega{+}D)^{-1}) = (\det(C\Omega{+}D))^k \Phi(\Omega)$$

where A, B, C, D are 2×2 matrices satisfying

1. $AB^T = BA^T$, $CD^T = DC^T$, $AD^T - BC^T = 1$
2. Entries of A, B, C, D are integers
3. Entries of C are integer multiples of N
4. $\det A - 1$ is integer multiple of N

We have an explicit algorithm for constructing $\Phi(t,u,v)$ and hence $g(t,u,v)$ and $f(k,l,m)$.

Given a computer we can calculate $f(k,l,m)$ for any k,l,m.

A prediction for the exact number of states with a given set of charges.

The result was guessed by Dijkgraaf, Verlinde and Verlinde for the $N = 1$ case.

Our work generalizes this to $N = 2, 3, 5, 7$.

Some sort of proof was given for the $N = 1$ case by Shih, Strominger, Yin (hep-th/0505094) by mapping this to the problem of calculating degeneracy of a rotating black hole in five dimensions.

Question: Can we generalize this proof to other values of N?

In our earlier construction we expressed Φ as a series expansion of the form

$$\Phi(t,u,v) = \sum_{k,l,m} \psi(k,l,m)\, e^{2\pi i(k\,t + l\,u + m\,v)}$$

$\psi(k,l,m)$: known coefficients

Now we have a different proposal for Φ in a product form:

$$\Phi(t,u,v) = \prod_{k,l,m} \left(1 - e^{2\pi i(k\,t + l\,u + m\,v)}\right)^{c(k,l,m)}$$

$c(k,l,m)$: known coefficients

$$\Phi(t,u,v) = \sum_{k,l,m} \psi(k,l,m)\, e^{2\pi i(k\,t+l\,u+m\,v)}$$

$$\Phi(t,u,v) = \prod_{k,l,m} \left(1 - e^{2\pi i(k\,t+l\,u+m\,v)}\right)^{c(k,l,m)}$$

This implies non-trivial relation between the $\psi(k,l,m)$'s and $c(k,l,m)$'s.

We have checked numerically that the first few terms in the expansion of these two expressions agree.

According to the second expression:

$$g(t,u,v) = \prod_{k,l,m} \left(1 - e^{2\pi i(k\,t + l\,u + m\,v)}\right)^{-c(k,l,m)}$$

This is easier to interprete in terms of a counting problem.

Thus there is hope that using this new expression we shall be able to interprete our proposal for $d(\vec{Q},\vec{M})$ as the result of a counting problem.

This may eventually lead to a proof of our conjecture.

Application of our results to black hole physics:

We know that for large charges our formula reproduces the entropy of a black hole carrying the same charges.

But since we have an exact formula, we can systematically calculate corrections to the entropy as a power series expansion in inverse power of charges.

Question: Can we reproduce these corrections from systematic computation of black hole entropy directly?

If so this would test our understanding of black hole physics beyond the leading approximation.