Uniqueness of the Foliation of Spherically Symmetric Static Spactimes by Flat Spacelike Hypersurfaces Corresponding to Freely Falling Observers

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- Foliation
- Foliation by Flat Spacelike Hyperurfaces
- Uniqueness of Flat Hypersurfaces

Foliation

- Splitting a space (S) into a sequence of subspaces (SS) such that each and every point of the space lies on one and only one of the subspaces is called a foliation.
- Codim of Foliation = dim(S) - dim(SS)
- Subspaces (SS) are called *hypersurfaces* if Codimension = 1

A hypersurface is flat if all the components of the Riemann Curvature Tensor are zero.

Consider Spherically Symmetric Static Spacetime metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}\,dr^2 - r^2 d\Omega^2$$

Where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

To obtain flat hypersurfaces solve

$$R^{i}{}_{jkl} = 0 \qquad (i, j, k, l = 1, 2, 3)$$

Instead: Writing unit tangent vector to the world-line of the observer falling freely from infinity as *n* and the the unit tangent vector to the hypersurface as *T*, we require that

$$n \bullet n = 1 = -T \bullet T, \quad T \bullet n = 0 \text{ ---- } \alpha$$

$$\frac{dt}{dr} = \pm \frac{e^{\frac{\lambda-\nu}{2}}\sqrt{k^2 - e^{-\lambda}}}{k}$$

We have flat Hypersurfaces if *k=1*.

Now Consider hypersurface

$$f(t, r, \theta, \phi) = 0$$

Considering spherical symmetry the above hypersurface in explicit form can be given as

$$t = F(r)$$

The induced metric (of hypersurfaces) is

$$ds_3^2 = -\left(e^{\lambda(r)} - e^{\nu(r)} F'^2\right) dr^2 - r^2 d\Omega^2$$

For the induced metric to be flat a necessary but not sufficient condition

i.e. $\text{Ricci Scalar}: \quad R = 0$

$$\Rightarrow$$

$$\frac{r\left(-\lambda' e^{\lambda} + \nu' e^{\nu} F'^2 + 2 e^{\nu} F' F''\right)}{\left(e^{\lambda} - e^{\nu} F'^2\right)^2} + \frac{1 - e^{\lambda} + e^{\nu} F'^2}{e^{\lambda} - e^{\nu} F'^2} = 0 \qquad (1)$$

Using the substitution

$$k^2(r) = \frac{1}{e^{\lambda(r)} - e^{\nu(r)} F'^2}$$

Equation (1) becomes

$$2rkk' + k^2 - 1 = 0,$$

and we have the solution

$$k^2(r) = 1 - \frac{c}{r} \quad (c \text{ is arbitrary constant})$$

The induced metric is then

$$ds_3^2 = -\frac{dr^2}{1-\frac{c}{r}} - r^2 d\Omega^2$$

The above metric is flat i.e.

$$R^1{}_{212} = R^1{}_{313} = R^2{}_{323} = 0 \quad \text{if} \quad c = 0$$

$$\therefore \qquad k(r) = 1.$$

The Flat Hypersurfaces are then given as:

$$t = F(r) = \int e^{\frac{\lambda - \nu}{2}} \sqrt{1 - e^{-\lambda}}\, dr.$$

The mean extrinsic curvature, ***K***, of these hypersurfaces is

$$K = e^{-\left(\frac{\nu+\lambda}{2}\right)}\left(\frac{\nu' e^{\nu}}{2\sqrt{1-e^{\nu}}} - \frac{2\sqrt{1-e^{\nu}}}{r}\right)$$

Thank You