Diffeomorphism + Weyl invariant scalar field theory coupled to two-dimensional gravity, on the disc, with insertion of arbitrary vertex operators.

Disc with marked pts. only admit hyperbolic metric of -ve. curvature

¿ Continuum limit of scalar field theory on Bethe lattices should be a good description. ?

Continuum limit

$m \to \infty$

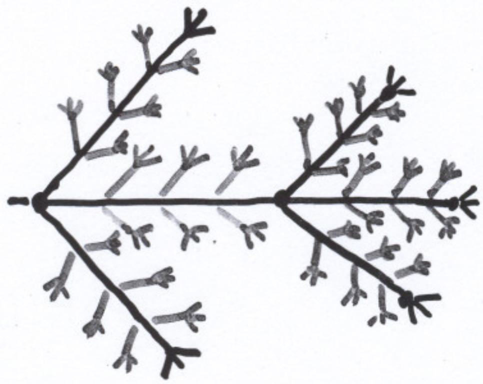
$$\lim_{m \to \infty} \pi = \lim_{m \to \infty} p^{1/m} = 1$$

Further :

- Random walk on Bethe lattice of coord. no. p $\xrightarrow{p \to 1}$ Brownian moti on hyperbolic plane

Green's fn. for Diffusion eqn. $\longrightarrow$ Green's fn. on hyperbolic plan

[Monthus & Texier (1996)]



- divide lattice spacing $\Delta r \equiv a$ into $(m-1)$ smaller units s.t. $a' = \frac{a}{m}$

- integrate out orange branche and rescale

Lattice corresponding to the (totally ramified) algebraic extension of $\mathbb{Q}_p \equiv K^{(m)}$

$$\mathbb{Q}_p \longrightarrow p$$

$$K^{(m)} \longrightarrow \pi = p^{1/m}$$

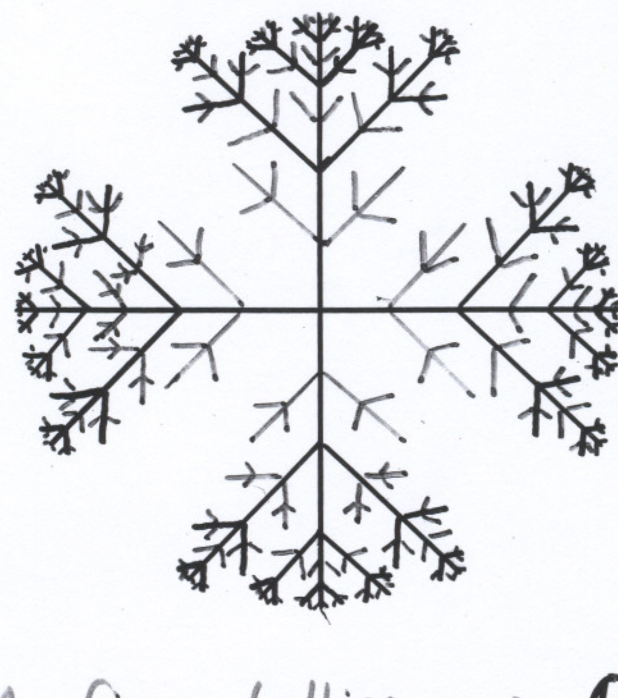
Introduce a lattice spacing Δr

$$p = 1 + \frac{\Delta r}{a}(N-1)$$

$$\Rightarrow \mathcal{N}(n) \cong \mathrm{vol}_N(R)$$

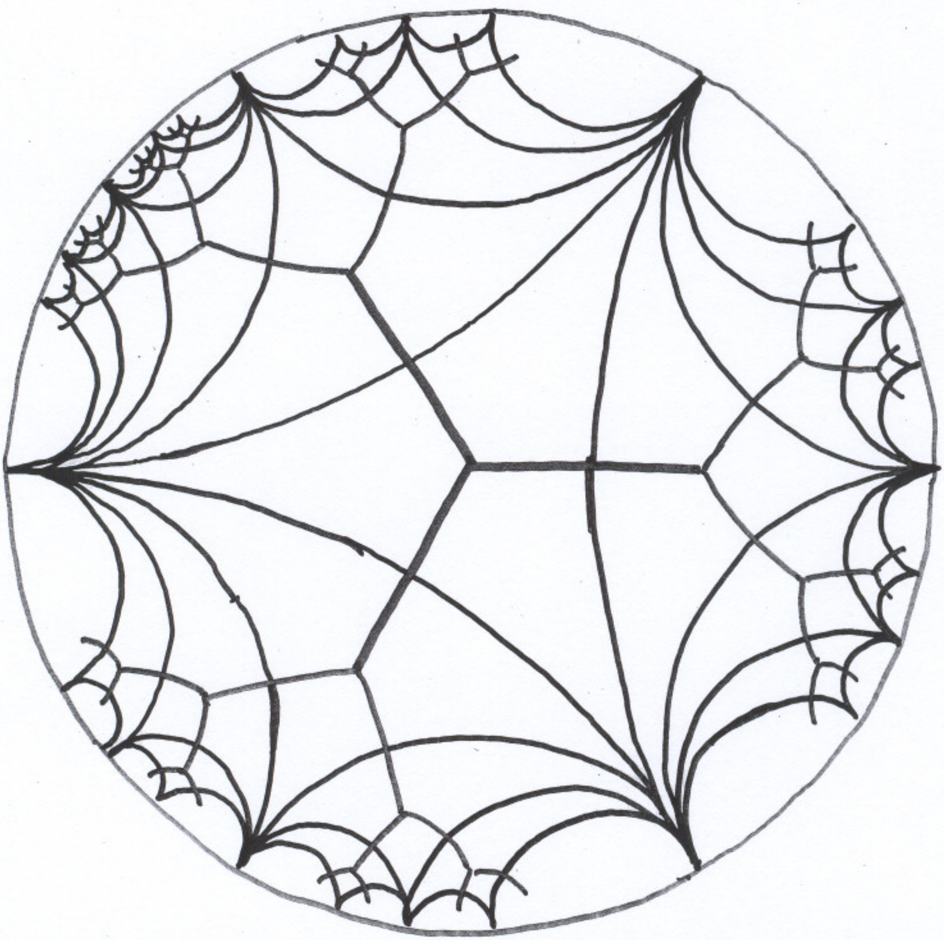
$$\text{w/} \quad \lim_{\substack{n \to \infty \\ \Delta r \to 0}} n \cdot \Delta r = R$$

$p \to 1^+$ LIMIT OF A BETHE LATTICE APPROXIMATES A CONTINUUM HYPERBOLIC SPACE.



A finer lattice $\rightsquigarrow$ $\mathbb{Q}_3[\sqrt{3}]$

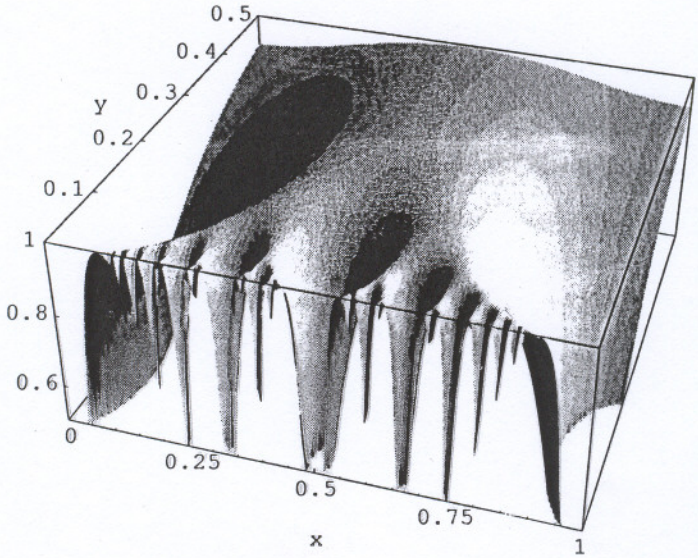
HYPERBOLIC TESSELLATION/ TILING

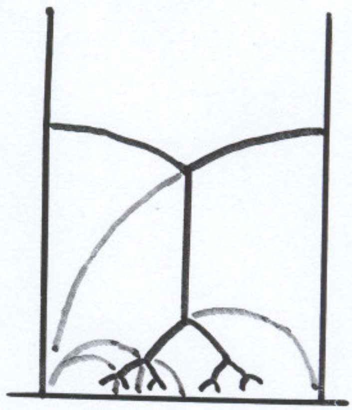


⑥

Sometimes related to Modular functions, number theory ...

There is a nice function on the UHP, whose absolute values give a potential for this discretisation.





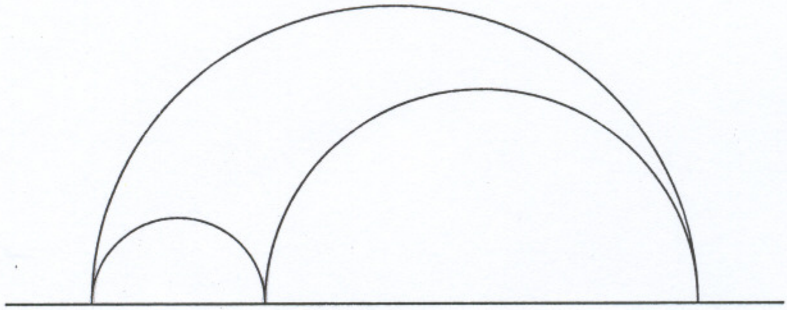
$$f(x,y) \sim y^{1/4} \left| e^{i\pi z/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right) \right|$$

Hyperbolic tessellation

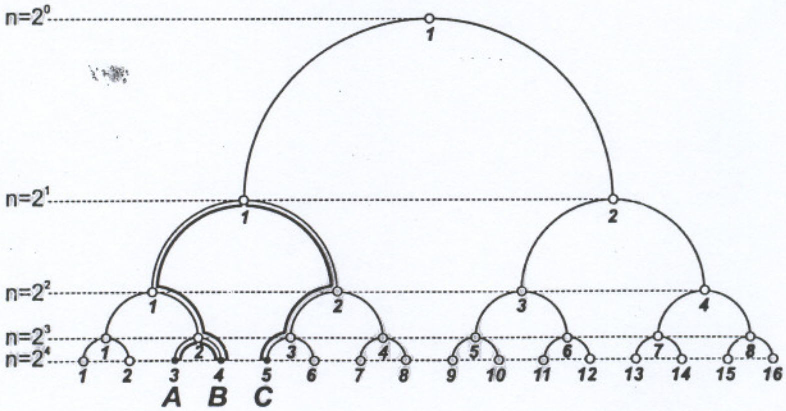
BETHE LATTICE CAN DISCRETISE <u>HYPERBOLIC SPACES</u>, e.g. OF DIMN 2.
UHP/DISC WITH THE POINCARÉ METRIC:

The Poincaré metric

$$ds^2 = (dx^2 + dy^2)/y^2$$



and its discretisation:



But, the no. of sites upto generation n from an origin grows as

$$\mathcal{N}(n) = 1 + (p+1) + (p+1)p + \cdots + (p+1)p^n$$

$$\overset{n\to\infty}{\sim} \; p^n = \exp.(n \ln p)$$

EXPONENTIAL GROWTH

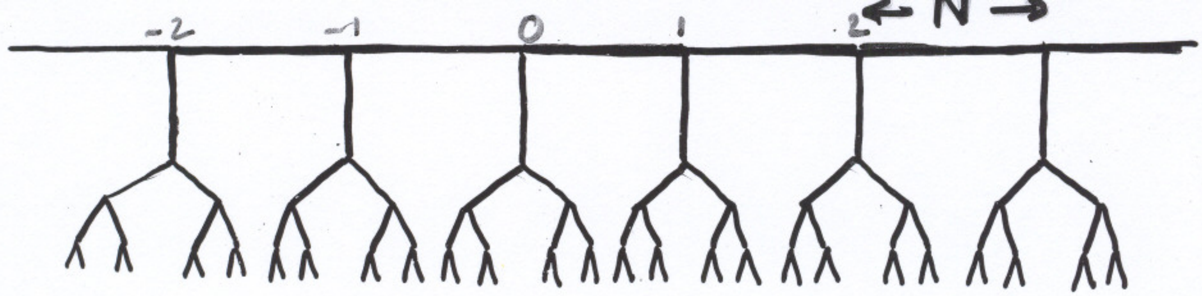
Formal dimension of Bethe lattice is ∞, if embedded in an Euclidean space.

In an N-dimensional Hyperbolic space

$$ds_N^2 = dr^2 + a^2 \sinh^2\left(\frac{r}{a}\right) d\Omega_{N-1}^2$$

curvature $= -1/a^2$

$$\underbrace{\mathrm{vol}_N(R)}_{\substack{\text{volume of a ball} \\ \text{of radius } R \gg a}} \sim a^N \Omega_{N-1} \exp.\left[\frac{R}{a}(N-1)\right]$$



The boundary of the tree $\mathcal{T}_p$ is $\mathbf{Q}_p$ (through the power series expansion

$$\xi = p^N(\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots),$$

$\xi_n \in \{0, 1, \cdots, p-1\}, \xi_0 \neq 0.$)

The usual string worldsheet is the strip which is conformally equivalent to the UHP = SL(2,**R**)/SO(2,**R**). SL(2,**R**) acts on the boundary of the worldsheet as a symmetry, (and SO(2) is its maximal compact subgroup).

The symmetry group of the boundary of the p-adic worldsheet is PGL(2,$\mathbf{Q}_p$).

$$\boxed{\mathrm{PGL}(2,\mathbf{Q}_p)/\mathrm{PGL}(2,\mathbf{Z}_p) = \mathcal{T}_p}$$

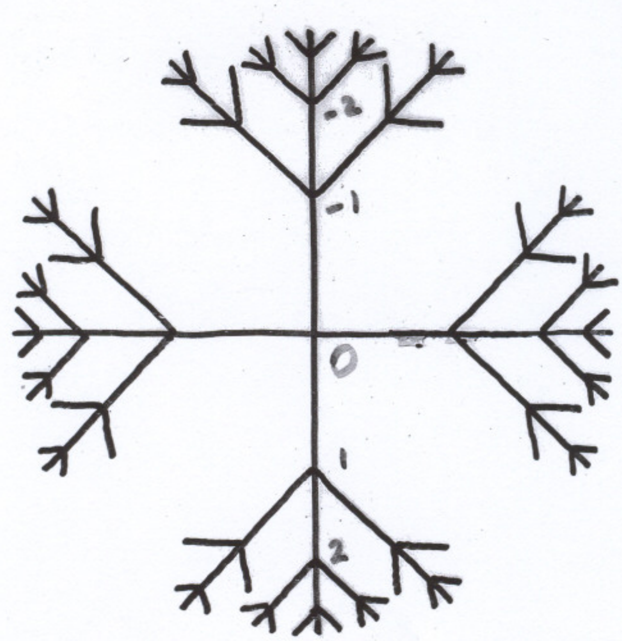
The ‘worldsheet’ of the p-adic string is *discrete*, but its boundary is a *continuum*.

p-adic string provides an *exotic discretisation* of the open string worldsheet.

WHY / HOW DOES THE p-ADIC STRING BECOME THE USUAL BOSONIC STRING IN THE LIMIT $p \to 1$?

CLUE FROM THE WORLDSHEET ?

THE 'WORLDSHEET' OF THE p-STRING IS AN INFINITE BETHE LATTICE, A TREE WITH COORDINATION NO. $= p$.



'Worldsheet' of the 3-adic string: Bethe lattice of coord. no. 3

There are also nontrivial solutions: these are again gaussian solitonic lumps.

Strategy:
We observe that $\star$-product of gaussians is again a gaussian:

$$Ae^{-a|z|^2} \star Be^{-b|z|^2} = \frac{AB}{1+ab\theta^2} \exp\left(\frac{a+b}{1+ab\theta^2}|z|^2\right).$$

Now make a gaussian ansatz:

$$\varphi(x^1, x^2) = A^2 \exp\left(-a\left[(x^1)^2 + (x^2)^2\right]\right)$$

and equate width a and amplitude A on both sides of eom.

$\Rightarrow$ Polynomial eqn. for p-adic

$\rightarrow$ Transcendental eqn. for $p \to 1$

Interpolating solitons

$$\text{p-soliton} \xrightarrow{\theta} \text{GMS}$$

DG(2004)

DG & Kawano (2004)

(12)

Let us deform the tachyon action of the p-adic string. Using the $\star$-product:

$$\mathcal{L}_{NC}^{(p)} = -\varphi \star p^{-\frac{1}{2}\Box}\varphi + \frac{1}{p+1}(\star\varphi)^{p+1},$$

Equation of motion in the noncommutative theory:

$$p^{-\frac{1}{2}\Box}\varphi = (\star\varphi)^p.$$

We will be interested only in the part dependent on x^1 and x^2.

The solutions, $\varphi = 0, 1$ describing constant configurations, are still solutions of the deformed eom.

A CURIOUS

'Continuum' limit: $p \to 1$!

Take $\boxed{p = 1 + \epsilon}$ and consider the limit $\epsilon \to 0$. We get,

$$\mathcal{L}^{(p\to 1)} = -\frac{1}{2}\varphi \Box \varphi - \frac{1}{2}\varphi^2 \left(\ln \varphi^2 - 1 \right).$$

This is exactly what one gets from a string field theory, **of the usual bosonic string!**

More precisely, it is the boundary string field theory (BSFT) action for the tachyon of the bosonic string, truncated to two derivatives! (Gerasimov & Shatashvili; Kutasov, Marino & Moore)

Introduce noncommutativity [Cornalba, Okuyama]:

$$\mathcal{L}_{NC}^{(p\to 1)} = -\frac{1}{2}\varphi \star \Box \varphi - \frac{1}{2}\varphi \star \varphi (\ln_\star(\varphi \star \varphi) - 1)$$

yields the equation of motion:

$$\Box \varphi + 2\varphi \star \ln_\star \varphi = 0.$$

$\Uparrow$

const. B-field

This is a solitonic m-brane.

Its tension (=energy/volume) is

$$\begin{aligned}\mathcal{T}_m &= \frac{p^2}{2g^2(p+1)}\left[\frac{2\pi p^{2p/(p-1)}\ln p}{p^2-1}\right]^{(25-m)/2}\\ &\equiv \frac{1}{2g_m^2}\frac{p^2}{p+1}.\end{aligned}$$

Ratio of tension $\frac{\mathcal{T}_m}{\mathcal{T}_{m-1}}$ is independent of m: as in ordinary string theory.

Can study the worldvolume theory on the soliton. A consistent truncation keeping only the tachyon is possible. There are also massless fields corresponding to translation zero modes.

p-Tachyon field behaves according to the conjectures of Sen. (DG & Sen, 2000)

Equation of motion:

$$\boxed{p^{-\frac{1}{2}\Box}\varphi = \varphi^p.}$$

The solutions to these are:

- The *constant* configuration $\boxed{\varphi(x) = 1.}$

  This is the perturbative solution — a space-filling D-25-brane.

  The *tension* of the brane is

  $$\mathcal{T}_{25} = -\mathcal{L}(\phi = 1) = \frac{1}{2g^2}\frac{p^2}{p+1}.$$

- The *constant* configuration $\boxed{\varphi(x) = 0.}$

  There are *no* perturbative excitation around this configuration. Energy vanishes in this configuration.
  Closed string vacuum?

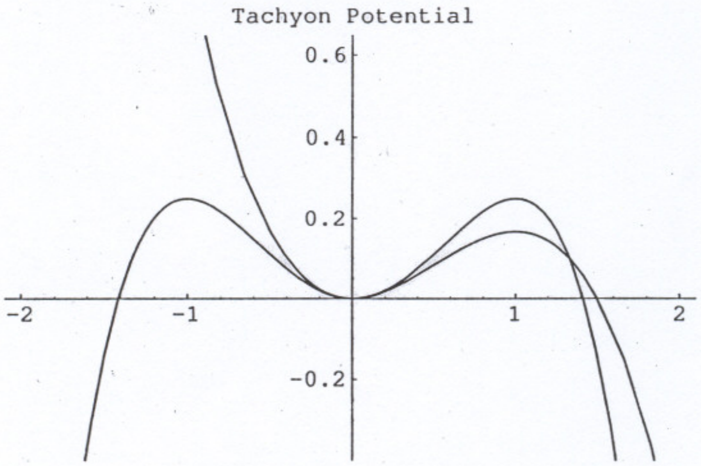
- Space(time) dependent solutions:

  Gaussian lump of every codimension.

Kinetic term in is non-standard and involves an infinite number of derivatives. This is a non-local theory.

Potential has a local minimum and two (respectively one) local maxima for odd (respectively even) integer p:

p-tachyon potential



Potential always has runaway pathological singularities, as in bosonic string.

The tachyon effective action (in spacetime) is:

$$
\begin{aligned}
\mathcal{L}^{(p)} &\simeq \frac{p}{p-1}\Big[-\frac{1}{2}Tp^{-\frac{1}{2}\Box-1}T \\
&\quad +\frac{p}{g^2(p+1)}\left(1+\frac{gT}{p}\right)^{p+1}-\frac{T}{g}\Big] \\
&\boxed{\simeq \frac{p^2}{g^2(p-1)}\left[-\frac{1}{2}\varphi p^{-\frac{1}{2}\Box}\varphi+\frac{1}{p+1}\varphi^{p+1}\right]}
\end{aligned}
$$

$T(x)$ is the tachyonic scalar and

$$
\varphi(x) = 1 + \frac{g}{p}T(x)
$$

Perturbative vacuum $T = 0$ correspond to $\varphi = 1$.

The key observation of *BFOW*: all the ingredients of the amplitude $\mathcal{A}_N$ have p-adic analogues.

Proposal: Define p-string theory as follows:

- replace the absolute value norm $|\cdot|$ of $\mathbf{R}$ in the integrand by the p-adic norm $|\cdot|_p$ of $\mathbf{Q}_p$:

$$|\cdot| \to |\cdot|_p,$$

- replace integrals over $\mathbf{R}$ by integrals over $\mathbf{Q}_p$ (using the latter's translationally invariant measure).

The consequence of this is amazing:
All N-point amplitudes for p-tachyons can now be computed exactly.
I.e., we know the effective field theory of the tachyonic scalar field of p-string theory *exactly*.

Any number in $\mathbf{Q}_p$ can be represented as a power series in p

$$\boxed{\xi = p^N \left(\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots\right)}$$

where, $\xi_n = \{0, 1, 2, \cdots, p-1\}$, $\xi_0 \neq 0$.

E.g. in $\mathbf{Q}_7$, $\frac{1}{2} = 4 + 3.7 + 3.7^2 + 3.7^3 + \cdots$.

This is like the Laurent series expansion.

Elements of $\mathbf{Q}_p$ with norm at most 1, form a maximal compact subring:

$$\mathbf{Z}_p = \{\xi \in \mathbf{Q}_p : \; |\xi|_p \leq 1\},$$

the ring of p-adic integers. Ordinary integers are a dense subset in it.

\end{digression}

Examples: $|5|_5 = \frac{1}{5}$, $|35|_5 = \frac{1}{5}$, $|250|_5 = \frac{1}{5^3} = \frac{1}{125}$,
$|6|_5 = 1$, $\left|\frac{2}{3}\right|_5 = 1$, $\left|\frac{2}{5}\right|_5 = 5$, $|96|_2 = \frac{1}{32}$.

$|q|_p$ satisfies all the properties of a norm. In fact there is a stronger triangle inequality:

$$|q_1 + q_2|_p \leq \max \left\{|q_1|_p, |q_2|_p\right\}$$

This norm is called *non-Archimedian* and leads to many strange properties:

- All triangles are *isosceles*.

- Any point in the *interior* of a disc is a *centre*.

- If $|q_1|_p < |q_2|_p$, then $|n\, q_1|_p < |q_2|_p$ for any $n \in \mathbf{Z}$.

Now recall that the field of real number $\mathbf{R}$ is obtained from $\mathbf{Q}$ adding to $\mathbf{Q}$ the *limit points* of Cauchy sequences. The limit points are determined by the absolute value norm.

Cauchy completion of $\mathbf{Q}$ by the p-adic norm leads to a new field of numbers $\mathbf{Q}_p$

\begin{digression}

Consider the *rational numbers* $\mathbf{Q}$. We are familiar with the *absolute value norm*. Now we will define different norms on this field.

1. Fix a prime number $p \in \{2, 3, 5, 7, 11, 13, \cdots\}$,

2. For any integer z, find the *highest* power of p that divides it. Call it $\mathrm{ord}_p z$.

   $\mathrm{ord}_p \sim$ logarithm: $\mathrm{ord}_p(z_1 z_2) = \mathrm{ord}_p z_1 + \mathrm{ord}_p z_1$.

3. For a rational number $q = \frac{z_1}{z_2}$:
   $\mathrm{ord}_p q = \mathrm{ord}_p z_1 - \mathrm{ord}_p z_2$.

4. The *p-adic norm* is:

$$|q|_p = \begin{cases} p^{-\mathrm{ord}_p q}, & q \neq 0 \\ 0, & q = 0 \end{cases}$$

p-STRING THEORY

à la Brekke-Freund-Olson-Witten (1987)

Recall tree level scattering amplitude of N on-shell tachyons of momenta k_i $(i = 1, \cdots, N)$, $k_i^2 = 2, \sum k_i = 0$ is:

$$\begin{aligned} \mathcal{A}_N(\{k_i\}) &= \int d\xi_1 \cdots d\xi_{N-3} \\ &\times \prod_{i=1}^{N-3} |\xi_i|^{k_N \cdot k_i} |1 - \xi_i|^{k_{N-1} \cdot k_i} \\ &\times \prod_{1 \le i < j \le N-3} \left| \xi_i - \xi_j \right|^{k_i \cdot k_j} . \end{aligned}$$

The integrals are over the real line **R** with its translationally invariant measure $d\xi$. Integrand involves absolute values of real numbers.

The 4-point amplitude can be computed exactly in terms of Euler beta function. But $\mathcal{A}_N$ for $N \geq 5$ cannot be evaluated analytically.

Plan

- A brief introduction to p-string theory
  - with a digression to arithmetic
- p-tachyon effective action & verification of Sen's Conjectures
- Non-commutative deformations Interpolating p-solitons
- A curious continuum limit

$$\boxed{p \to 1}$$

- Why / How $p \to 1$?

STRINGS vs. p-STRINGS

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Work in progress

& • w/ T. Kawano, Nucl. Phys. B710 (2005) 577

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