FOKKER PLANCK KOLMOGOROV EQUATION FOR fBm: Derivation & Analytical Solution

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OBJECTIVES

1) Derive FPK Equation for

$$dx_i = f_i(\underset{\sim}{x}, t)\,dt + g_{i\alpha}\,dW_\alpha^H \quad \left(\begin{matrix} i=1,\dots,n \\ \alpha=1,\dots,r \end{matrix}\right)$$

2) Prove the Liouville's theorem for these systems.

3) Prove the Fluctuation-Dissipation theorem for fBm.

4) Generalize step 3

5) Give some physically relevant examples.

0. PRELIMINARIES

DEFINITION: A zero mean Gaussian proces W_t^H with

$$E[W_t^H W_s^H] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right], \quad H \in (0,1)$$

is called fBm.

Some properties of the fBm.

i) $E[(W_t^H)^2] = t^{2H}, \quad W_0^H = 0$

ii) $E[(W_t^H - W_s^H)^2] = |t-s|^{2H}$

iii) $C(s) \equiv E[(W_{t+1}^H - W_t^H)(W_{t+s+1}^H - W_{t+s}^H)]$

$$= \frac{1}{2}\left[|s+1|^{2H} + |s-1|^{2H} - 2|s|^{2H}\right]$$

$C(s) < 0 \quad H \in (0, 1/2)$

$C(s) = 0 \quad H = 0$

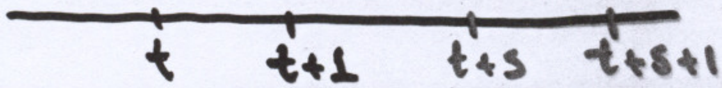
$C(s) > 0 \quad H \in (1/2, 1)$

iv) Although W_t^H is continuos it is no where differentiable

v) fBm is a self-similar process, i.e,

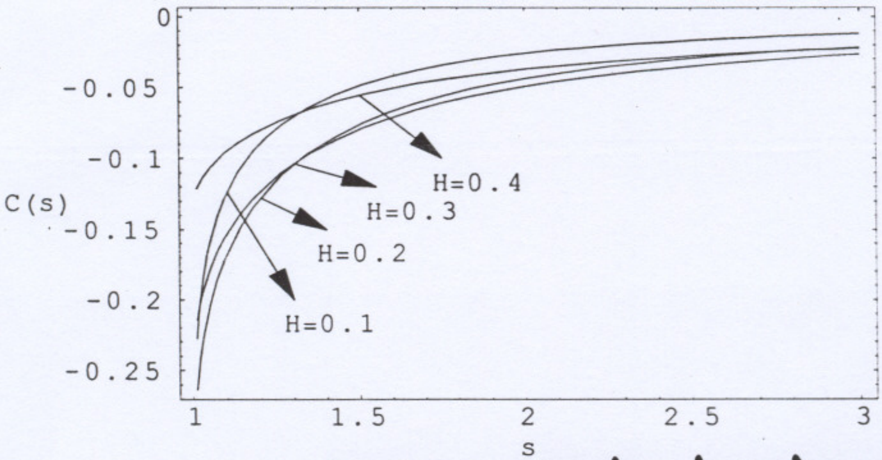
$$\frac{1}{a^H} W_{at}^H \overset{d}{=} W_t^H \qquad a > 0$$

vi) $E[dW_t^H] = 0$

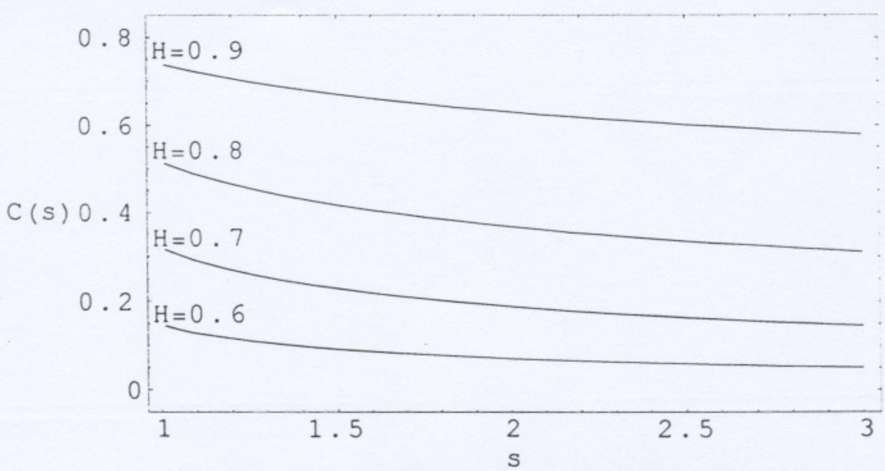


$$C(s) = E\left[\left(W^H_{t+1} - W^H_t\right)\left(W^H_{t+s+1} - W^H_{t+s}\right)\right]$$

$$= \frac{1}{2}\left[|s+1|^{2H} + |s-1|^{2H} - 2|s|^{2H}\right]$$



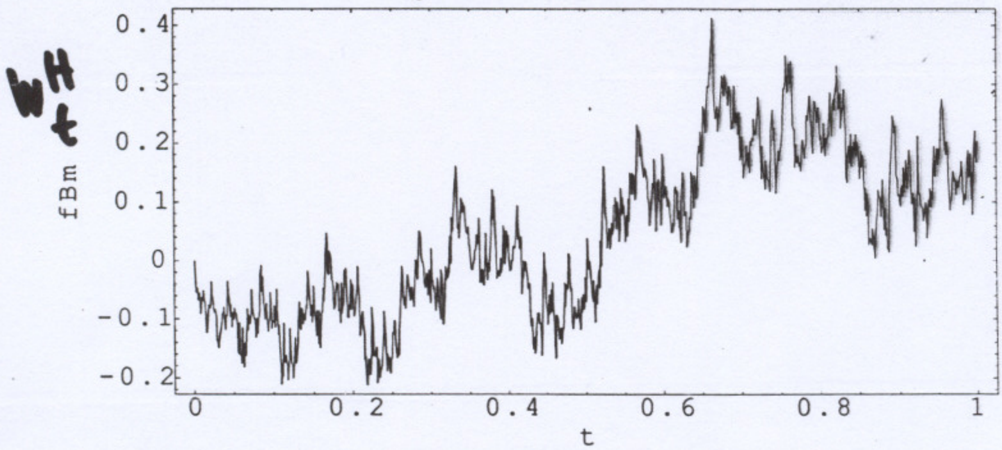
Antipersistent behaviour



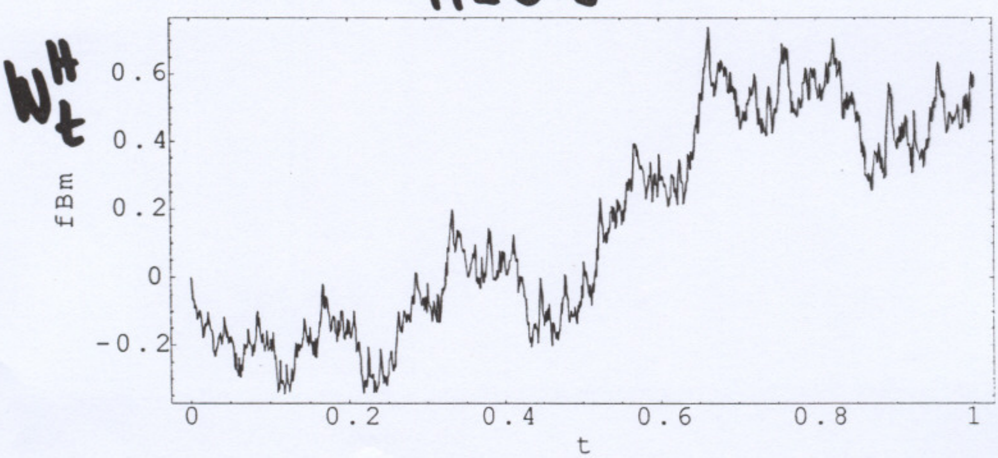
Persistent behaviour

$$E[W_t^H W_s^H] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

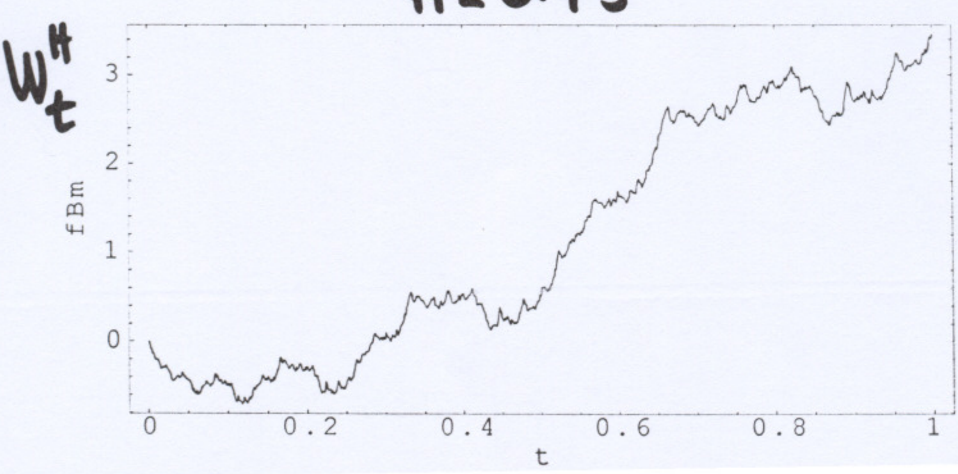
H=0.35



H=0.5



H=0.75



$$\int W^H \, dW^H = ?$$

(5)

$$df(t, W^H) = \left(\frac{\partial f}{\partial t} + H t^{2H-1} \frac{\partial^2 f}{\partial W^{H2}}\right) dt + \frac{\partial h}{\partial W^H} dW^H$$

$$f = (W^H)^2$$

$$\int W^H \, dW^H = \frac{(W^H)^2}{2} - \frac{1}{2} t^{2H}$$

$t \geq 0$

4. APPLICATIONS

JAYNES-CUMMING MODEL WITH fBm

$$dS_1 = -S_2\,dt - d_1 S_1\,dt + g_1\,dW_1^H$$

$$dS_2 = (S_1 + S_3 E)\,dt - d_2 S_2\,dt + g_2\,dW_2^H$$

$$dS_3 = -S_2 E\,dt - d_3 S_3\,dt + g_3\,dW_3^H$$

$$dE = F\,dt \quad , \quad dF = (-\nu^2 E + \alpha S_1)\,dt$$

$d_1 = d_1(S_2, S_3, E, F, t)$, $d_2 = d_2(S_1, S_3, E, F, t)$, $d_3 = d_3(S_1, S_2, E, F, t)$

$g_1 = g_1(S_2, S_3, E, F, t)$, $g_2 = g_2(S_1, S_3, E, F, t)$, $g_3 = g_3(S_1, S_2, E, F, t)$

S_1, S_2, S_3 : dimensionless Bloch vector, E: Electric field

α, ν : constant parameters.

Associated FPK Equation is

$$\frac{\partial p}{\partial t} - S_2 \frac{\partial p}{\partial S_1} + (S_1 + S_3 E)\frac{\partial p}{\partial S_2} - S_2 E \frac{\partial p}{\partial S_3} + F \frac{\partial p}{\partial E}$$

$$+ (-\nu^2 E + \alpha S_1)\frac{\partial p}{\partial F} + \frac{\partial}{\partial S_1}(-d_1 S_1 p) + \frac{\partial}{\partial S_2}(-d_2 S_2 p)$$

$$+ \frac{\partial}{\partial S_3}(-d_3 S_3 p) - H t^{2H-1}\left(g_1^2 \frac{\partial^2 p}{\partial S_1^2} + g_2^2 \frac{\partial^2 p}{\partial S_2^2} + g_3^2 \frac{\partial^2 p}{\partial S_3^2}\right) = 0.$$

When $\quad d_i = H t^{2H-1} g_i^2 \qquad i = 1, 2, 3$

A normalizable solution is

$$p = K e^{-\beta(S_1^2 + S_2^2 + S_3^2)}$$

$$I_1 = S_1^2 + S_2^2 + S_3^2 ; \; I_2 = \alpha S_3 - \alpha S_1 E + \tfrac{1}{2}\nu^2 E^2 + \tfrac{1}{2} F^2 .$$

Collisionless Plasma

$$dx_1 = x_2 x_3\,dt - 2a x_1 d_1\,dt + g_1\,dW_1^H$$

$$dx_2 = \left(-\frac{a}{b} x_1 x_3\right)dt - 2(b+c)x_2 d_2\,dt + g_2\,dW_2^H$$

$$dx_3 = \left(-\frac{ac}{bd} x_1, x_2\right)dt - 2d\,x_3 d_3\,dt + g_3\,dW_3^H$$

a, b, c, d are constants. $d_i = d_i(x_1, \hat{x}_i, x_3)$ $g_i = g_i(.., \hat{x}_i.$

Associated FPK equation is

$$\frac{\partial p}{\partial t} + x_2 x_3 \frac{\partial p}{\partial x_1} - \frac{a}{b} x_1 x_3 \frac{\partial p}{\partial x_2} - \frac{ac}{bd} x_1 x_2 \frac{\partial p}{\partial x_3} + \frac{\partial}{\partial x_1}(-2a x_1 d_1 p)$$
$$+ \frac{\partial}{\partial x_2}(-2(b+c)d_2 x_2 p) + \frac{\partial}{\partial x_3}(-2d\,x_3 d_3) - Ht^{2H-1}\Big(g_1^2 \frac{\partial^2 p}{\partial x_1^2}$$
$$+ g_2^2 \frac{\partial^2 p}{\partial x_2^2} + g_3^2 \frac{\partial^2 p}{\partial x_3^2}\Big) = 0$$

It has a normalizable solution of the form

$$p = K\,e^{-(a x_1^2 + (b+c)x_2^2 + c x_3^2)}$$

Provided that

$$d_i = Ht^{2H-1}(g_i)^2$$

Notice that VF $\underset{\sim}{c}$ has the following conserved quantities

$$I_1 = a x_1^2 + b x_2^2, \quad I_2 = c x_2^2 + d x_3^2$$