Chern-Simons Theory on S^1-Bundles: Abelianisation and q-deformed Yang-Mills Theory

George Thompson

ICTP, P.O. Box 586, 34100 Trieste, Italy.

Talk presented in Islamabad

Work done in collaboration with Matthias Blau: hep-th/0601068

The object of interest to us is the path integral,

$$Z_{k,\,G}[M] = \int DA \exp \frac{ik}{4\pi} \int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 \right)$$

The path integral is over the connections on $E \longrightarrow M$ a G bundle.
We will consider special classes of 3-manifold M namely those that are themselves $U(1)$ bundles, of degree p, over Σ_g a Riemann surface of genus g. Denote these by $M_{(g,p)}$.

The steps involved in solving the theory on $U(1)$ bundles are:

1. Decomposing the gauge field

The steps involved in solving the theory on $U(1)$ bundles are:

1. Decomposing the gauge field
2. Fixing the gauge (partially)

The steps involved in solving the theory on $U(1)$ bundles are:

1. Decomposing the gauge field
2. Fixing the gauge (partially)
3. Diagonalising

The steps involved in solving the theory on $U(1)$ bundles are:

1. Decomposing the gauge field
2. Fixing the gauge (partially)
3. Diagonalising
4. Pushing the calculations down to the base Σ_g.

$$Z_{k,\,G}[M] = \Lambda \frac{e^{4\pi i p \Phi_0}}{|W|} \sum_{r \in \mathbb{Z}_p^{\mathbf{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \mathrm{Tr}\left(p\phi^2 + 4\pi r \phi\right)$$

Λ is independent of p, $T_{S^1}(\phi) = \det_{\mathfrak{k}}\left(1 - \mathrm{Ad}\, e^{\phi}\right)$ is the Ray-Singer Torsion on the circle and $\Phi_0 = \dim G/48$.

One can perform the Gaussian integrals and/or the sums but this is the nicest form so I leave it like this.

Background

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.
   - One gets an invariant for all G and k (even for G non-compact)

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.
   - One gets an invariant for all G and k (even for G non-compact)
   - LMO have extracted from this a mathematically well defined theory of 'finite type' invariants.

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.
   - One gets an invariant for all G and k (even for G non-compact)
   - LMO have extracted from this a mathematically well defined theory of 'finite type' invariants.
   - One can also consider knot invariants (or link invariants)

$$Z_{k,\,G}[M, \gamma, R] = \langle \mathrm{Tr}_R\, P \exp \oint_\gamma A \rangle$$

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.
   - One gets an invariant for all G and k (even for G non-compact)
   - LMO have extracted from this a mathematically well defined theory of 'finite type' invariants.
   - One can also consider knot invariants (or link invariants)

$$Z_{k,\,G}[M, \gamma, R] = \langle \mathrm{Tr}_R \, P \, \exp \oint_\gamma A \rangle$$

2. Exact results also exist.

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.
   - One gets an invariant for all G and k (even for G non-compact)
   - LMO have extracted from this a mathematically well defined theory of 'finite type' invariants.
   - One can also consider knot invariants (or link invariants)

$$Z_{k,\,G}[M, \gamma, R] = \langle \mathrm{Tr}_R \, P \, \exp \oint_\gamma A \rangle$$

2. Exact results also exist.
   - Surgery prescription combined with CFT techniques allow one to determine the Chern-Simons invariants.

Background

1. The path integral gives (framed) 3-manifold invariants.
   - Usually one follows a perturbative approach $A \to A/\sqrt{k}$ and expands in large k.
   - One gets an invariant for all G and k (even for G non-compact)
   - LMO have extracted from this a mathematically well defined theory of 'finite type' invariants.
   - One can also consider knot invariants (or link invariants)

$$Z_{k,\,G}[M,\gamma,R] = \langle \mathrm{Tr}_R\, P\, \exp \oint_\gamma A \rangle$$

2. Exact results also exist.
   - Surgery prescription combined with CFT techniques allow one to determine the Chern-Simons invariants.
   - But there are very few exact (i.e. non-perturbative) path integral computations of Chern-Simons theory, the exceptions being on manifolds of the form $\Sigma_g \times S^1$. The technique employed was Abelianisation of the non-Abelian theory.

We are reconsidering the path integral on $M_{(g,p)}$

- To ask if the method of Abelianisation can be generalised from trivial ($p = 0$) to non-trivial circle bundles.

We are reconsidering the path integral on $M_{(g,p)}$

- To ask if the method of Abelianisation can be generalised from trivial ($p = 0$) to non-trivial circle bundles.
- As Chern-Simons theory on general Seifert manifolds has recently been studied by Beasley and Witten **hep-th/0503126** using the method of non-Abelian localisation one would like toknow if the diagonalisation procedure yields results that are manifestly equivalent or comparable to those of BW?

We are reconsidering the path integral on $M_{(g,p)}$

- To ask if the method of Abelianisation can be generalised from trivial ($p = 0$) to non-trivial circle bundles.
- As Chern-Simons theory on general Seifert manifolds has recently been studied by Beasley and Witten **hep-th/0503126** using the method of non-Abelian localisation one would like toknow if the diagonalisation procedure yields results that are manifestly equivalent or comparable to those of BW?
- Chern-Simons theory on Lens spaces $L(p, 1) = M_{(0,p)}$ has also recently appeared in the context of black hole partition function calculations via topological string theory **hep-th/0211098** and the methods of Abeliansiation were used to argue that this theory is equivalent to a "q-deformed" two-dimensional Yang-Mills theory.

Geometric Set-Up

The $M_{(g,p)}$ are themselves principal $U(1)$ bundles $U(1) \longrightarrow M_{(g,p)} \xrightarrow{\pi} \Sigma_g$ over 2-dimensional surfaces Σ_g of genus g and first Chern (or Euler) class $-p \in \mathbb{Z}$.
Let κ be a connection on the principal $U(1)$-bundle $M_{(g,p)}$ (a globally defined real-valued 1-form on the total space of the bundle), and denote by K the fundamental vector field on $M_{(g,p)}$, i.e. the generator of the $U(1)$-action.

- A connection κ is characterised by $\iota_K \kappa = 1$

- A connection κ is characterised by $\iota_K \kappa = 1$
- and the equivariance condition $L_K \kappa = 0$ where $L_K = \{d , \iota_K\}$ is the Lie derivative in the K direction. These two conditions imply that $\iota_K d\kappa = 0$, i.e. that the curvature 2-form $d\kappa$ of κ is horizontal (as it should be).

- A connection κ is characterised by $\iota_K \kappa = 1$
- and the equivariance condition $L_K \, \kappa = 0$ where $L_K = \{d \, , \, \iota_K\}$ is the Lie derivative in the K direction. These two conditions imply that $\iota_K d\kappa = 0$, i.e. that the curvature 2-form $d\kappa$ of κ is horizontal (as it should be).
- In local coordinates $\kappa = d\theta + a$ where θ is a fibre coordinate, $0 \leq \theta < 1$, and $a = a_i \, dx^i$ is a local representative on Σ_g of the connection κ on $M_{(g,p)}$.

- A connection κ is characterised by $\iota_K \kappa = 1$
- and the equivariance condition $L_K\, \kappa = 0$ where $L_K = \{d\,,\, \iota_K\}$ is the Lie derivative in the K direction. These two conditions imply that $\iota_K d\kappa = 0$, i.e. that the curvature 2-form $d\kappa$ of κ is horizontal (as it should be).
- In local coordinates $\kappa = d\theta + a$ where θ is a fibre coordinate, $0 \leq \theta < 1$, and $a = a_i\, dx^i$ is a local representative on Σ_g of the connection κ on $M_{(g,p)}$.
- Since $M_{(g,p)}$ has degree p, we may choose κ so that

$$d\kappa = p\, \pi^*(\omega), \quad \int_{\Sigma_g} \omega = 1.$$

- A connection κ is characterised by $\iota_K \kappa = 1$
- and the equivariance condition $L_K \, \kappa = 0$ where $L_K = \{d \, , \, \iota_K\}$ is the Lie derivative in the K direction. These two conditions imply that $\iota_K d\kappa = 0$, i.e. that the curvature 2-form $d\kappa$ of κ is horizontal (as it should be).
- In local coordinates $\kappa = d\theta + a$ where θ is a fibre coordinate, $0 \leq \theta < 1$, and $a = a_i \, dx^i$ is a local representative on Σ_g of the connection κ on $M_{(g,p)}$.
- Since $M_{(g,p)}$ has degree p, we may choose κ so that

$$d\kappa = p \, \pi^*(\omega), \quad \int_{\Sigma_g} \omega = 1.$$

- With the above choices we see that

$$\kappa \wedge d\kappa = p \, d\theta \wedge \pi^*(\omega)$$

is nowhere vanishing as required providing that the $U(1)$ bundle is non-trivial, that is providing $p \neq 0$.

- A connection κ is characterised by $\iota_K \kappa = 1$
- and the equivariance condition $L_K \kappa = 0$ where $L_K = \{d \, , \iota_K\}$ is the Lie derivative in the K direction. These two conditions imply that $\iota_K d\kappa = 0$, i.e. that the curvature 2-form $d\kappa$ of κ is horizontal (as it should be).
- In local coordinates $\kappa = d\theta + a$ where θ is a fibre coordinate, $0 \leq \theta < 1$, and $a = a_i \, dx^i$ is a local representative on Σ_g of the connection κ on $M_{(g,p)}$.
- Since $M_{(g,p)}$ has degree p, we may choose κ so that

$$d\kappa = p \, \pi^*(\omega), \quad \int_{\Sigma_g} \omega = 1.$$

- With the above choices we see that

$$\kappa \wedge d\kappa = p \, d\theta \wedge \pi^*(\omega)$$

  is nowhere vanishing as required providing that the $U(1)$ bundle is non-trivial, that is providing $p \neq 0$.
- For later use we note that

$$\int_M \kappa \wedge d\kappa = p \int_{\Sigma_g} \omega = p \quad .$$

Decomposition

- As both K and κ are nowhere vanishing we have that $\kappa \wedge \iota_K$ and $(1 - \kappa \wedge \iota_K)$ are projection operators, corresponding to the decomposition

$$T^*M = T^*_\kappa(M) \oplus T^*_H(M), \quad T^*_\kappa(M) \approx \Omega^0(M)$$

into forms along the κ direction and those which are horizontal.

Decomposition

- As both K and κ are nowhere vanishing we have that $\kappa \wedge \iota_K$ and $(1 - \kappa \wedge \iota_K)$ are projection operators, corresponding to the decomposition

$$T^*M = T^*_\kappa(M) \oplus T^*_H(M), \quad T^*_\kappa(M) \approx \Omega^0(M)$$

into forms along the κ direction and those which are horizontal.

- Concretely, for $\alpha \in \Omega^1(M, \mathbb{R})$ one has $\alpha = \alpha_\kappa + \alpha_H$ with

$$\alpha_\kappa = \kappa \wedge \iota_K \, \alpha \in \Omega^1_\kappa(M, \mathbb{R}), \quad \alpha_H = (1 - \kappa \wedge \iota_K) \, \alpha \in \Omega^1_H(M, \mathbb{R}).$$

Decomposition

- As both K and κ are nowhere vanishing we have that $\kappa \wedge \iota_K$ and $(1 - \kappa \wedge \iota_K)$ are projection operators, corresponding to the decomposition

$$T^*M = T^*_\kappa(M) \oplus T^*_H(M), \quad T^*_\kappa(M) \approx \Omega^0(M)$$

  into forms along the κ direction and those which are horizontal.
- Concretely, for $\alpha \in \Omega^1(M, \mathbb{R})$ one has $\alpha = \alpha_\kappa + \alpha_H$ with

$$\alpha_\kappa = \kappa \wedge \iota_K \, \alpha \in \Omega^1_\kappa(M, \mathbb{R}), \quad \alpha_H = (1 - \kappa \wedge \iota_K) \, \alpha \in \Omega^1_H(M, \mathbb{R}).$$

- Likewise we can decompose connections on vector bundles E over M, thought of as elements of $\Omega^1(M, \mathfrak{g})$,

$$A = A_\kappa + A_H \equiv \phi \, \kappa + A_H.$$

  Since $\phi \in \Omega^0(M, \mathfrak{g})$ we can think of it as a section of the adjoint bundle $E = M \times \mathfrak{g}$.

The Action Becomes

$$\mathrm{Tr} \int_{M_{(g,p)}} \left(A_H \wedge \kappa \wedge \mathcal{L}_\phi A_H + 2\phi\, \kappa \wedge dA_H + \phi^2\, \kappa \wedge d\kappa \right)$$

here

$$\mathcal{L}_\phi B = L_K B + [\phi, B].$$

Things are looking good! The action still has a cubic term but as far as the horizontal fields are concerned we only have Gaussian integration to perform.

Gauge Transformations

Under infinitesimal gauge transformations $\delta A = d_A \Lambda$ is

$$\delta\phi = \mathcal{L}_\phi \Lambda \ \ ,$$

with

$$\mathcal{L}_\phi = L_K + [\phi, \ ].$$

Gauge Choice (partial)

- Having singled out a particular component of the gauge field A_κ it is tempting to impose the gauge condition $A_\kappa = 0 = \phi$. However, this is not possible since Wilson loops along the fibres of $M_{(g,p)} \to \Sigma_g$ are gauge invariant and non-trivial.

Gauge Choice (partial)

- Having singled out a particular component of the gauge field A_κ it is tempting to impose the gauge condition $A_\kappa = 0 = \phi$. However, this is not possible since Wilson loops along the fibres of $M_{(g,p)} \to \Sigma_g$ are gauge invariant and non-trivial.
- Instead we impose, the allowed condition,

$$L_K A_\kappa = 0 \Leftrightarrow L_K \phi = \iota_K \, d\, \phi = 0 \quad .$$

Gauge Choice (partial)

- Having singled out a particular component of the gauge field A_κ it is tempting to impose the gauge condition $A_\kappa = 0 = \phi$. However, this is not possible since Wilson loops along the fibres of $M_{(g,p)} \to \Sigma_g$ are gauge invariant and non-trivial.
- Instead we impose, the allowed condition,

$$L_K A_\kappa = 0 \Leftrightarrow L_K \phi = \iota_K \, d\, \phi = 0 \ \ .$$

- This gauge condition, $L_K \phi = 0$, tells us that ϕ is a $U(1)$-invariant section of E. Equivalently, it can therefore be regarded as a section of the (trivial) adjoint bundle V over Σ_g.

Gauge Choice (partial)

- Having singled out a particular component of the gauge field A_κ it is tempting to impose the gauge condition $A_\kappa = 0 = \phi$. However, this is not possible since Wilson loops along the fibres of $M_{(g,p)} \to \Sigma_g$ are gauge invariant and non-trivial.
- Instead we impose, the allowed condition,

$$L_K A_\kappa = 0 \Leftrightarrow L_K \phi = \iota_K \, d\, \phi = 0 \quad .$$

- This gauge condition, $L_K \phi = 0$, tells us that ϕ is a $U(1)$-invariant section of E. Equivalently, it can therefore be regarded as a section of the (trivial) adjoint bundle V over Σ_g.
- So ϕ 'only depends on Σ_g'.

Diagonalization

- Let T be some maximal torus of G and $\mathfrak{t}$ the corresponding Cartan subalgebra, with $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$.

Diagonalization

- Let T be some maximal torus of G and $\mathfrak{t}$ the corresponding Cartan subalgebra, with $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$.
- There are still S^1 independent gauge transformations $L_K\Lambda = 0$,

$$\delta\phi = [\phi, \Lambda]$$

  and one checks that $L_K\delta\phi = 0$ so the partial gauge is maintained.

Diagonalization

- Let T be some maximal torus of G and $\mathfrak{t}$ the corresponding Cartan subalgebra, with $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$.
- There are still S^1 independent gauge transformations $L_K \Lambda = 0$,
$$\delta\phi = [\phi, \Lambda]$$
and one checks that $L_K \delta\phi = 0$ so the partial gauge is maintained.
- Use the residual conjugation symmetry to impose
$$\phi^{\mathfrak{k}} = 0.$$

Diagonalization

- Let T be some maximal torus of G and $\mathfrak{t}$ the corresponding Cartan subalgebra, with $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$.
- There are still S^1 independent gauge transformations $L_K \Lambda = 0$,

$$\delta\phi = [\phi, \Lambda]$$

  and one checks that $L_K \delta\phi = 0$ so the partial gauge is maintained.
- Use the residual conjugation symmetry to impose

$$\phi^{\mathfrak{k}} = 0.$$

- So ϕ is in the Cartan sub-algebra (i.e. chargeless w.r.t the $U(1)$'s) and we are almost done, or are we?

- No the diagonal gauge, $\phi^{\mathfrak{k}} = 0$, is junk. Here is the standard counter example.

- No the diagonal gauge, $\phi^{\mathfrak{k}} = 0$, is junk. Here is the standard counter example.
- Set $\Sigma_g = S^2$, $G = SU(2)$ so that $\mathfrak{g} = \mathbb{R}^3$ and pick ϕ so that $\phi^2 = 1$ Then $\phi : S^2 \to S^2$ as the identity map. This map then has winding number one. If we could smoothly conjugate this ϕ into diagonal form we would find that we would have smoothly deformed ϕ to $\widetilde{\phi} = \sigma_3$ which is a map $\widetilde{\phi} : S^2 \to pt = (0, 0, 1)$. Since $\widetilde{\phi}$ has winding number zero this is clearly impossible.

Gauge Invariant Winding Number and Singular Gauge Transformations

- A gauge invariant winding number is

$$n(\phi, A, \Sigma_g) = -\int_{\Sigma_g} \mathrm{Tr}\left(\frac{1}{2\pi}\phi \wedge F_A + \frac{1}{16\pi}\phi\, d\phi \wedge d\phi\right)$$

with $n(\phi, 0, \Sigma_g) = \int_{\Sigma_g} \phi^*(\omega)$ the winding number of the map and $n(\phi^h, A^h, \Sigma_g) = n(\phi, A, \Sigma_g)$ its gauge invariance.

Gauge Invariant Winding Number and Singular Gauge Transformations

- A gauge invariant winding number is

$$n(\phi, A, \Sigma_g) = -\int_{\Sigma_g} \mathrm{Tr}\left(\frac{1}{2\pi}\phi \wedge F_A + \frac{1}{16\pi}\phi\, d\phi \wedge d\phi\right)$$

with $n(\phi, 0, \Sigma_g) = \int_{\Sigma_g} \phi^*(\omega)$ the winding number of the map and $n(\phi^h, A^h, \Sigma_g) = n(\phi, A, \Sigma_g)$ its gauge invariance.

- Applying this to our favourite map,

$$\text{winding\#} = n(\phi, 0, \Sigma_g) = n(\sigma_3, 0^h, \Sigma_g) = \frac{1}{2\pi}\int_{\Sigma_g} da = c_1(L)$$

with $a = -\mathrm{Tr}\, \sigma_3 h^{-1} dh$.

- Moral of the story is that if we insist on imposing the diagonal gauge then one must do so using singular gauge transformations and then in turn we produce non-trivial $U(1)$ bundles. If the map has winding number t we will get a $U(1)$ bundle with first Chern class t (we have split $su(2) \to L \oplus L^{-1}$). Consequently we should sum over all $U(1)$ bundles on Σ_g!

- Moral of the story is that if we insist on imposing the diagonal gauge then one must do so using singular gauge transformations and then in turn we produce non-trivial $U(1)$ bundles. If the map has winding number t we will get a $U(1)$ bundle with first Chern class t (we have split $su(2) \rightarrow L \oplus L^{-1}$). Consequently we should sum over all $U(1)$ bundles on Σ_g!
- From the 3-dimensional perspective over all T-bundles that one gets by pull back from Σ_g. Since the pull-back $\pi^* M_{(g,p)}$ of the $U(1)$-bundle $M_{(g,p)} \xrightarrow{\pi} \Sigma_g$ to the total space $M_{(g,p)}$ is (tautologically) trivial, $\pi^* M_{(g,p)} = M_{(g,p)} \times U(1)$, the pull-back of the p-th power of any line bundle on Σ_g to $M_{(g,p)}$ is trivial. Thus the pull-backs of line bundles from Σ_g to $M_{(g,p)}$ are of finite order. All torsion (finite order p) bundles on $M_{(g,p)}$ arise in this way, so that it is precisely these bundles that we should sum over in the path integral.

- Moral of the story is that if we insist on imposing the diagonal gauge then one must do so using singular gauge transformations and then in turn we produce non-trivial $U(1)$ bundles. If the map has winding number t we will get a $U(1)$ bundle with first Chern class t (we have split $su(2) \rightarrow L \oplus L^{-1}$). Consequently we should sum over all $U(1)$ bundles on Σ_g!
- From the 3-dimensional perspective over all T-bundles that one gets by pull back from Σ_g. Since the pull-back $\pi^* M_{(g,p)}$ of the $U(1)$-bundle $M_{(g,p)} \xrightarrow{\pi} \Sigma_g$ to the total space $M_{(g,p)}$ is (tautologically) trivial, $\pi^* M_{(g,p)} = M_{(g,p)} \times U(1)$, the pull-back of the p-th power of any line bundle on Σ_g to $M_{(g,p)}$ is trivial. Thus the pull-backs of line bundles from Σ_g to $M_{(g,p)}$ are of finite order. All torsion (finite order p) bundles on $M_{(g,p)}$ arise in this way, so that it is precisely these bundles that we should sum over in the path integral.
- Sum over all line bundles L of finite order $L^{\otimes p} = M_{(g,p)} \otimes \mathbb{C}$.

Torsion Bundles

- Consider $G = SU(2)$ and $T = U(1)$: a line bundle L on Σ_g has first Chern class $c_1(L) = r[\omega]$, so that $\pi^*(L)$ has first Chern class

$$c_1(\pi^*(L)) = r\,[\pi^*(\omega)] = \frac{r}{p}\,[d\kappa] \ .$$

Torsion Bundles

- Consider $G = SU(2)$ and $T = U(1)$: a line bundle L on Σ_g has first Chern class $c_1(L) = r[\omega]$, so that $\pi^*(L)$ has first Chern class

$$c_1(\pi^*(L)) = r\,[\pi^*(\omega)] = \frac{r}{p}\,[d\kappa] \quad .$$

- Thus the pull-back connection may be taken to be

$$\mathrm{A} = 2\pi \frac{r}{p}\kappa \quad .$$

Torsion Bundles

- Consider $G = SU(2)$ and $T = U(1)$: a line bundle L on Σ_g has first Chern class $c_1(L) = r[\omega]$, so that $\pi^*(L)$ has first Chern class

$$c_1(\pi^*(L)) = r\,[\pi^*(\omega)] = \frac{r}{p}\,[d\kappa] \ .$$

- Thus the pull-back connection may be taken to be

$$\mathrm{A} = 2\pi\frac{r}{p}\kappa \ .$$

- This connection has holonomy in the S^1 direction of $M_{(g,p)}$,

$$\exp\,(i\oint \mathrm{A}) = \exp\,(2\pi i\frac{r}{p}) \in \mathbb{Z}_p$$

and captures the torsion.

- The curvature 2-form F_H becomes

$$F_H = dA_H + d\mathsf{A} = dA_H + 2\pi \frac{r}{p} d\kappa$$

and the path integral should include a summation over $r = 0, \ldots, p-1$.

- The curvature 2-form F_H becomes

$$F_H = dA_H + d\mathrm{A} = dA_H + 2\pi \, \frac{r}{p} \, d\kappa$$

and the path integral should include a summation over $r = 0, \ldots, p-1$.

- This argument generalises to higher rank. Normalising the component fields by expanding ϕ and A_H in a basis of simple roots,

$$\phi = \sum_{i=1}^{\mathrm{rk}} \phi^i \, \alpha_i, \quad A_H = \sum_{i=1}^{\mathrm{rk}} A^i \, \alpha_i \quad ,$$

F_H is

$$F_H = \sum_{i=1}^{\mathrm{rk}} \left(dA_H^i + 2\pi \, \frac{r^i}{p} \, d\kappa \right) \, \alpha_i \quad .$$

The Calculation (partly done)

- The action is now

$$\int_M \mathrm{Tr}\left(A_H^{\mathfrak{t}} \wedge dA_H^{\mathfrak{t}} + A_H^{\mathfrak{k}} \wedge \kappa \wedge \mathcal{L}_\phi A_H^{\mathfrak{k}} + 2\phi\, \kappa \wedge F_H^{\mathfrak{t}} + \phi^2\, \kappa \wedge d\kappa\right)$$

The Calculation (partly done)

- The action is now

$$\int_M \mathrm{Tr}\left(A_H^{\mathfrak{t}} \wedge dA_H^{\mathfrak{t}} + A_H^{\mathfrak{k}} \wedge \kappa \wedge \mathcal{L}_\phi A_H^{\mathfrak{k}} + 2\phi\, \kappa \wedge F_H^{\mathfrak{t}} + \phi^2\, \kappa \wedge d\kappa\right)$$

- The charged gauge fields enter only through a Gaussian integration and so we can perform the path integral to get a determinant

$$\left(\mathrm{Det}\left(*\,\kappa \wedge\ i\mathcal{L}_\phi\right)_{\Omega^1_H(M,\mathfrak{k})}\right)^{-1/2}$$

The Calculation (partly done)

- The action is now

$$\int_M \mathrm{Tr}\left(A_H^{\mathfrak{t}} \wedge dA_H^{\mathfrak{t}} + A_H^{\mathfrak{k}} \wedge \kappa \wedge \mathcal{L}_\phi A_H^{\mathfrak{k}} + 2\phi\, \kappa \wedge F_H^{\mathfrak{t}} + \phi^2\, \kappa \wedge d\kappa\right)$$

- The charged gauge fields enter only through a Gaussian integration and so we can perform the path integral to get a determinant

$$\left(\mathrm{Det}\left(*\,\kappa \wedge\ i\mathcal{L}_\phi\right)_{\Omega^1_H(M,\mathfrak{k})}\right)^{-1/2}$$

- There is also a (charged) ghost contribution which is

$$\mathrm{Det}\left(i\mathcal{L}_\phi\right)_{\Omega^0(M,\mathfrak{t})}$$

The Calculation (partly done)

- The action is now

$$\int_M \mathrm{Tr}\left(A^{\mathfrak{t}}_H \wedge dA^{\mathfrak{t}}_H + A^{\mathfrak{k}}_H \wedge \kappa \wedge \mathcal{L}_\phi A^{\mathfrak{k}}_H + 2\phi\, \kappa \wedge F^{\mathfrak{t}}_H + \phi^2\, \kappa \wedge d\kappa\right)$$

- The charged gauge fields enter only through a Gaussian integration and so we can perform the path integral to get a determinant

$$\left(\mathrm{Det}\left(*\,\kappa \wedge\ i\mathcal{L}_\phi\right)_{\Omega^1_H(M,\mathfrak{k})}\right)^{-1/2}$$

- There is also a (charged) ghost contribution which is

$$\mathrm{Det}\left(i\mathcal{L}_\phi\right)_{\Omega^0(M,\mathfrak{t})}$$

- This ratio of determinants, though almost unity, gives us three things

1. (The absolute value) The Ray-Singer torsion of S^1 (with respect to the flat connection $2\pi i \phi d\theta$), raised to one half of the Euler characteristic of Σ_g, $T_{S^1}(\phi)^{\chi(\Sigma_g)/2}$.

1. (The absolute value) The Ray-Singer torsion of S^1 (with respect to the flat connection $2\pi i\phi d\theta$), raised to one half of the Euler characteristic of Σ_g, $T_{S^1}(\phi)^{\chi(\Sigma_g)/2}$.
2. (The phase) The famous shift $k \to k + c_{\mathfrak{g}}$

1. (The absolute value) The Ray-Singer torsion of S^1 (with respect to the flat connection $2\pi i\phi d\theta$), raised to one half of the Euler characteristic of Σ_g, $T_{S^1}(\phi)^{\chi(\Sigma_g)/2}$.
2. (The phase) The famous shift $k \to k + c_{\mathfrak{g}}$
3. (The phase) The phase $\Phi_0 = \frac{1}{48} \dim G$

1. (The absolute value) The Ray-Singer torsion of S^1 (with respect to the flat connection $2\pi i\phi d\theta$), raised to one half of the Euler characteristic of Σ_g, $T_{S^1}(\phi)^{\chi(\Sigma_g)/2}$.
2. (The phase) The famous shift $k \rightarrow k + c_{\mathfrak{g}}$
3. (The phase) The phase $\Phi_0 = \frac{1}{48}\dim G$

- Only those $A_H^{\mathfrak{t}}$ that are not constant on the S^1, $L_K A_H^{\mathfrak{t}} \neq 0$, appear in the first term of the action (and they appear no where else). Integrating these out gives rise to a determinant which is precisely cancelled by a similar (chargeless) ghost determinant.

Two Dimensional Theory

- Having integrated out all the $\mathfrak{k}$-valued fields as well as all the $\mathfrak{t}$-valued modes which are not $U(1)$ invariant, the Chern-Simons path integral essentially reduces to the path integral of an Abelian 2-dimensional gauge theory on Σ_g,

$$Z_k[M_{(g,p)}, G] \;\sim\; \mathrm{e}^{4\pi i p \Phi_0} \int D\phi \, DA_H^{\mathfrak{t}} \; T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \; . \\ . \; \exp\left(i \, \frac{k + c_{\mathfrak{g}}}{4\pi} \, S_M \right) \; .$$

The action is

$$S_M = \int_M \mathrm{Tr} \left(2\phi \, \kappa \wedge F_H + \phi^2 \, \kappa \wedge d\kappa \right),$$

- Since the path integral is only over invariant modes, we can push the action S_M down to Σ_g,

$$S_M \to S_\Sigma[A_H, \phi] = \frac{k + c_\mathfrak{g}}{4\pi} \int_{\Sigma_g} \mathrm{Tr}\left(2\phi\, F_H + p\phi^2 \omega\right) \ ,$$

where $A_H = A_H^\mathfrak{t}$ and $\phi = \phi^\mathfrak{t}$.

- Since the path integral is only over invariant modes, we can push the action S_M down to Σ_g,

$$S_M \rightarrow S_\Sigma[A_H, \phi] = \frac{k + c_{\mathfrak{g}}}{4\pi} \int_{\Sigma_g} \mathrm{Tr}\left(2\phi\, F_H + p\phi^2\omega\right) \ ,$$

where $A_H = A_H^{\mathfrak{t}}$ and $\phi = \phi^{\mathfrak{t}}$.

- We must sum over all torus on M. We should therefore sum over all allowed values of $r^i = 0, \ldots, p-1$. But how does the path integral know that $r^i = 0$ is the same as $r^i = p$? Shifting the r^i by multiples of p, $r^i \rightarrow r^i + p\gamma^i$, $\gamma^i \in \mathbb{Z}$ is tantamount to shifting F_H by an element $2\pi\gamma = 2\pi\gamma^i\alpha_i$ of the integral lattice $I = 2\pi\mathbb{Z}[\alpha_i]$ of G and happily the transformation

$$F_H \rightarrow F_H + 2\pi\, d\kappa\, \gamma \qquad \phi \rightarrow \phi - 2\pi\gamma \ .$$

is a symmetry of the theory.

Reduction to Finite Dimensional Integrals

- The path integral over $A^{\mathfrak{t}}_H$ imposes a (delta function) condition on ϕ, namely

$$\iota_K d(\kappa\,\phi) = 0 \ \ .$$

This delta function constraint on ϕ together with the gauge condition imply that ϕ is actually constant,

$$d\phi = 0.$$

Reduction to Finite Dimensional Integrals

- The path integral over $A_H^{\mathfrak{t}}$ imposes a (delta function) condition on ϕ, namely

$$\iota_K d(\kappa\,\phi) = 0 \quad .$$

This delta function constraint on ϕ together with the gauge condition imply that ϕ is actually constant,

$$d\phi = 0.$$

- With ϕ constant we have that

$$\int_M \mathrm{Tr}\ \kappa \wedge d\kappa\,\phi^2 = p\ \mathrm{Tr}\ \phi^2$$

- The partition function has reduced to the finite-dimensional integral

$$Z_k[M_{(g,p)}, G] \sim e^{4\pi i p \Phi_0} \sum_{r \in \mathbb{Z}_p^{\mathsf{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \, . \\ . \, \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \mathrm{Tr}\left(p\,\phi^2 + 4\pi r\,\phi\right)$$

- The partition function has reduced to the finite-dimensional integral

$$Z_k[M_{(g,p)},G] \;\sim\; e^{4\pi i p\Phi_0} \sum_{r\in\mathbb{Z}_p^{\mathsf{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \,. \\ .\; \exp i\frac{k+c_{\mathfrak{g}}}{4\pi}\,\mathrm{Tr}\left(p\,\phi^2 + 4\pi r\,\phi\right)$$

- Fix the remaining real normalisation constant, Λ, by comparison with the known normalisation for $p=0$.

- The partition function has reduced to the finite-dimensional integral

$$Z_k[M_{(g,p)}, G] \;\sim\; \mathrm{e}^{4\pi i p \Phi_0} \sum_{r \in \mathbb{Z}_p^{\mathsf{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} \,. \\ . \; \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \,\mathrm{Tr}\left(p\,\phi^2 + 4\pi r\,\phi\right)$$

- Fix the remaining real normalisation constant, Λ, by comparison with the known normalisation for $p = 0$.
- The formulae above are invariant under the action of the Weyl group W which is part of the original gauge group. We thus need to divide by the "volume" (or mod out by the action) of W.

- The partition function has reduced to the finite-dimensional integral

$$Z_k[M_{(g,p)}, G] \sim e^{4\pi i p \Phi_0} \sum_{r\in\mathbb{Z}_p^{\mathbf{rk}}} \int_{\mathfrak{t}} T_{S^1}(\phi)^{\chi(\Sigma_g)/2} .$$

$$. \exp i \frac{k + c_{\mathfrak{g}}}{4\pi} \mathrm{Tr}\left(p\,\phi^2 + 4\pi r\,\phi\right)$$

- Fix the remaining real normalisation constant, Λ, by comparison with the known normalisation for $p = 0$.
- The formulae above are invariant under the action of the Weyl group W which is part of the original gauge group. We thus need to divide by the "volume" (or mod out by the action) of W.
- The Ray-Singer torsion has zeros at the boundary of the Weyl chamber, which means that for genus $g > 1$ the integrals diverge. The way around this is to regularise by giving a small mass term to the connection, while preserving the residual $U(1)^{\mathbf{rk}}$ invariance.

Acknowledgements

I would like to thank the organizers of this meeting for the possibility to be here. My thanks especially to Faheem Hussain and Riazuddin.