Geometry and Symmetry in General Relativity

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1 Introduction

M is a 4-dimensional Hausdorff connected smooth manifold

g is a smooth Lorentz metric on M $(-,+,+,+)$

(M, g) is a space-time.

∇ is the Levi-Civita connection from g with Christoffel symbols Γ^a_{bc}

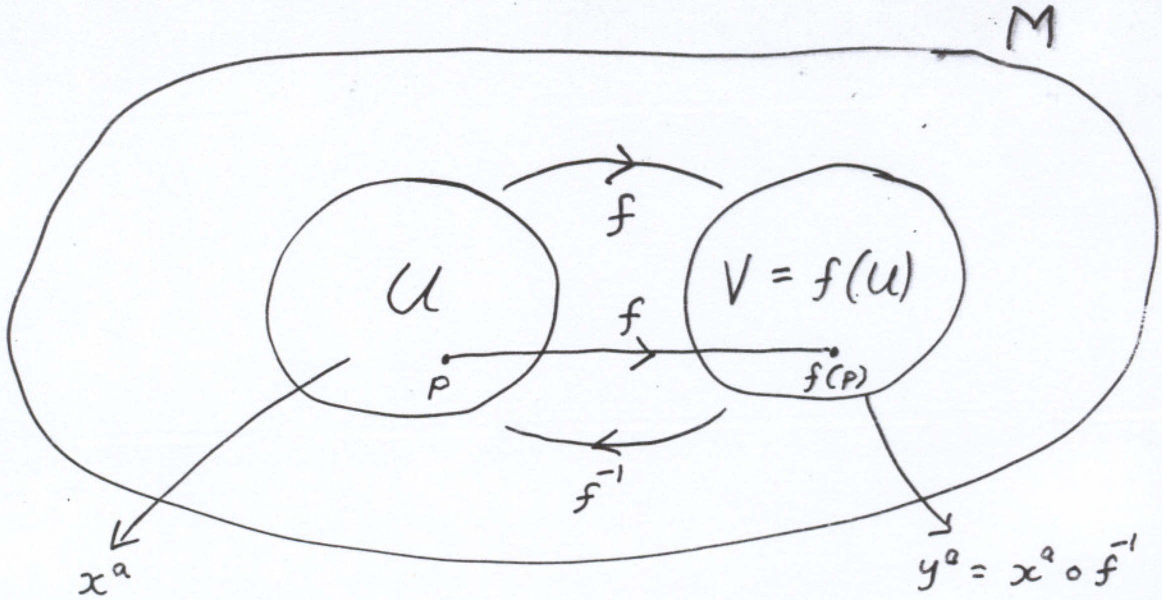
Riem is curvature tensor from ∇ $(R^a{}_{bcd})$

Ricc is Ricci tensor from ∇ (R_{ab})

$;$ covariant derivative $\quad$ $,$ partial derivative

$\mathcal{L}$ Lie derivative

2 The Geometry of Symmetry



U is open coordinate domain in M with coordinates x^a and f is a bijective map $f : U \to V$ $(= f(U))$ so $f^{-1} : V \to U$.

The map f is smooth so f is a local diffeomorphism.

So $y^a = x^a \circ f^{-1}$ is a coordinate system on V.

Let T be a tensor on M.

Define f to be a local symmetry of T if the components of T at p in the system x^a equal the components of T at $f(p)$ in the system y^a for each $p \in U$. $(\Leftrightarrow f^* T = T)$

Now let X be a <u>smooth vector field on M</u>. A <u>curve</u> $x^a(t)$ in some coordinate domain of M is an integral curve of X <u>starting from $p \in M$</u> if $\frac{dx^a}{dt} = X^a$ and if $x^a(0) = p$.

<u>Theorem</u> There exists an open neighbourhood U of p and $\varepsilon > 0$ such that for any $q \in U$ there is an integral curve $x^a(t)$ of X which starts from q and which is defined for each $t \in (-\varepsilon, \varepsilon)$.

So, for each $q \in U$ and for each $t \in (-\varepsilon, \varepsilon)$ we can "move" q along the integral curve of X, starting from q, a (curve) parameter distance t along this curve, to a point, say, q'.

Thus, for this $t \in (-\varepsilon, \varepsilon)$ each $q \in U$ is moved a parameter distance t in this way and gives rise to a map $\phi_t : U \to V \, (= \phi_t(U))$ where $\phi_t(q) = q'$. These maps, for the choices of U and t, are called the <u>local flows</u> or <u>local diffeomorphisms</u> of X.

So let us define a <u>symmetry of a Tensor T</u> as a vector field X such that each local flow of X is a local symmetry of T.

Then the statement that X is a symmetry of T is equivalent (by definition) to the statement that $\varphi_t^* T = T$ for each local flow φ_t of X. This is then equivalent to the condition that

$$\mathcal{L}_X T = 0 \tag{1}$$

[One could change the symmetry condition $f^* T = T$ to, for example, $f^* T = \lambda T$ for some function $\lambda : \mathcal{U} \to \mathbb{R}$, or to some other well-defined geometrical restriction on f]

The condition (1) is sometimes more useful than the relations $\varphi_t^* T = T$.

Perhaps the most important symmetry in general relativity (or in differential geometry) occurs when $T = g$. Thus we study the equation $\mathcal{L}_X g = 0$ (Killing's equation).

An equivalent form of it is

$$X_{a;b} + X_{b;a} = 0 \iff X_{a;b} = F_{ab} \left(= -F_{ba}\right) \quad (2)$$

Note: if $p \in M$ and $X(p) \neq 0$, choose coordinates x^a so that $X^a = (1, 0, 0, 0) = \delta^a_1$. Then (2) is equivalent to $\frac{\partial g_{ab}}{\partial x^1} = 0$ [and the integral curves of X are of the form $t \to (a+t, b, c, d)$ with $a, b, c, d \in \mathbb{R}$].

The skew-symmetric tensor F is called the Killing bivector.

Note that the condition $\phi_t^* g = g$ on the local flows of X is that ϕ_t be a local isometry of (M, g).

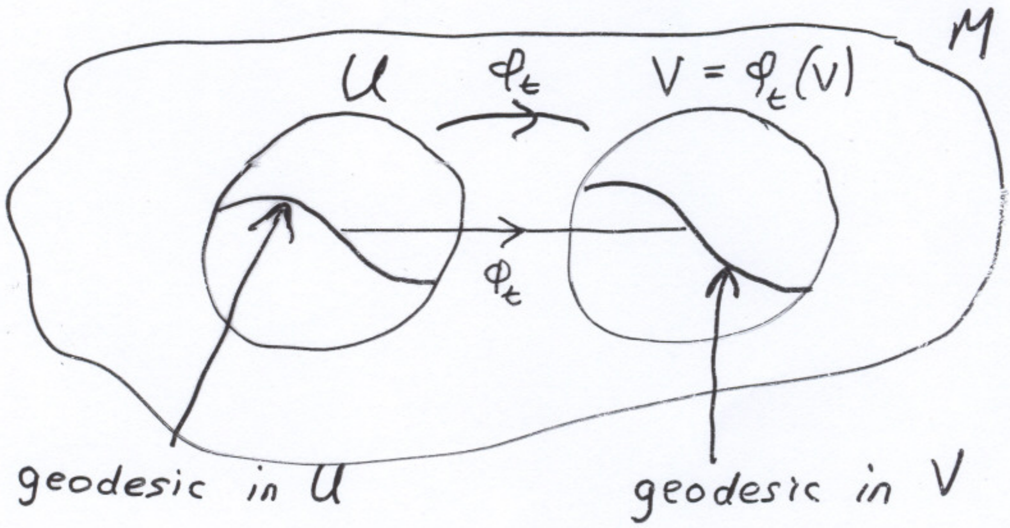
Other Symmetries are;

(i) When each ϕ_t satisfies $\phi_t^* g = \sigma g$ for $\sigma : U \to \mathbb{R}$ (i.e ϕ_t a conformal map)

Then $\mathcal{L}_X g = \psi g \qquad \psi : M \to \mathbb{R}$.

and X is a conformal vector field.

(ii) When each ϕ_t preserves (local) geodesics.



Then X is called a projective vector field

If, in addition, each ϕ_t also preserves affine parameters, the X is called affine.

Notation

Let $K(M)$, $H(M)$, $C(M)$, $A(M)$, $P(M)$ denote the Lie algebras of Killing, homothetic, conformal, affine and projective vector fields on M.

3 Symmetry Orbits.

Let $X_1, \cdots X_k \in K(M)$. Let their associated local flows be $\phi_t^1 \cdots \phi_t^k$. Let $p \in M$ and (assuming the following are defined) consider the following "movement" of p;

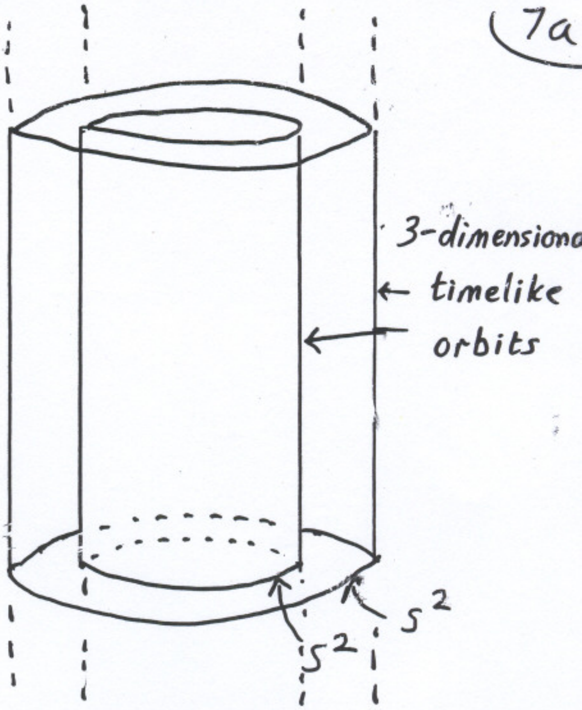
$$p \to \phi_{t_1}^1(p) \longrightarrow \phi_{t_2}^2\left(\phi_{t_1}^1(p)\right) \longrightarrow \cdots\cdots$$

All points which can be "reached" in this way from p make up the <u>orbit through p</u> and associated with $K(M)$.

These orbits are <u>submanifolds</u> of M and <u>integral manifolds</u> of $K(M)$. That is the tangent space to the orbit through p is the vector space $K(M)_p = \{X(p) : X \in K(M)\}$.

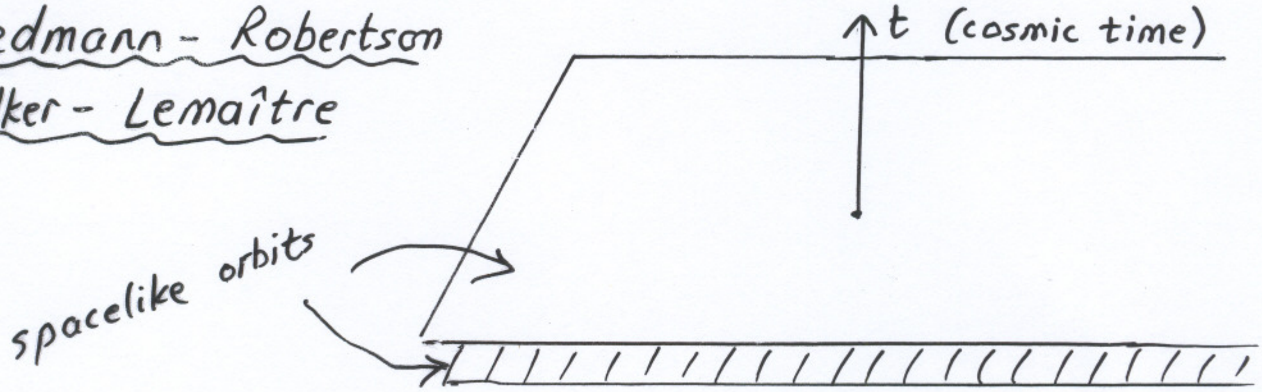
<u>Note</u> The dimension of $K(M)_p$ may change with p and so cannot necessarily use Fröbenius theory.

Examples



Schwarzschild
or Reissner–Nordstrom.

Friedmann–Robertson–Walker–Lemaître

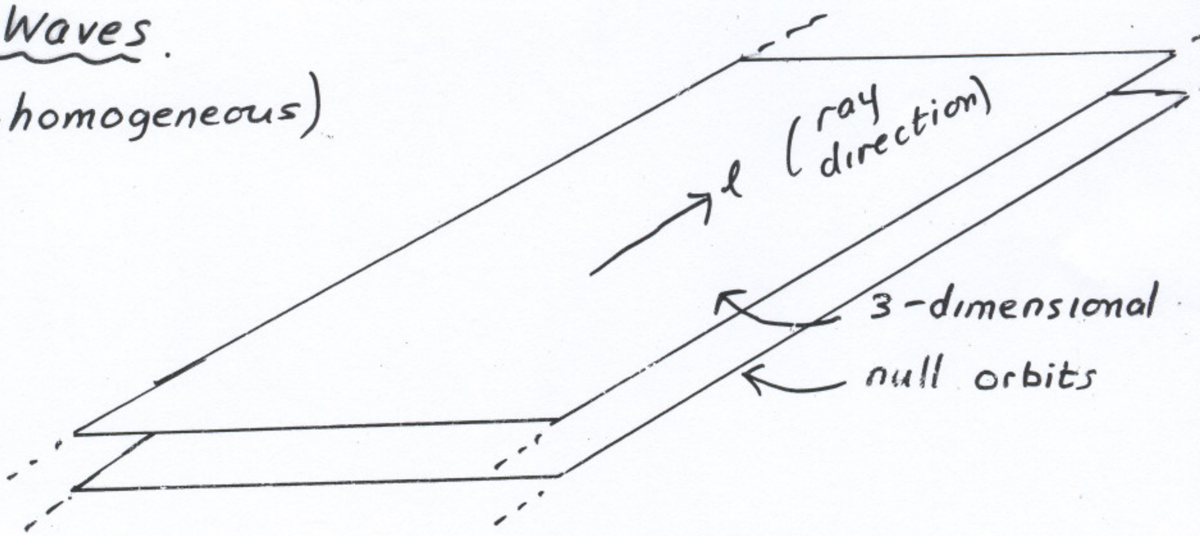


Einstein Static Universe.

(M is a single orbit – homogeneous case)

Plane Waves.

(non-homogeneous)



Example.

If $X \in K(M)$ then $\mathcal{L}_X R_{ab} = 0$.

Suppose $X(p) = 0$. Then, at p,

$$R_{ac} F^c{}_b + R_{bc} F^c{}_a = 0 \qquad (3)$$

Case 1 $F(p)$ simple. This means, at p,

$F_{ab} = r_a s_b - s_a r_b$ (r, s span blade of $F(p)$)

Case 2 $F(p)$ non-simple. This means, at p,

$$F_{ab} = \alpha(t_a z_b - z_a t_b) + \beta(x_a y_b + y_a x_b) \quad (\alpha, \beta \in \mathbb{R})$$

Here there are two orthogonal canonical blades of $F(p)$ spanned by t and z and by x and y.

Then it follows from (3) that if $F(p)$ is simple, each non-zero vector in the blade of $F(p)$ is an eigenvector of R_{ab} with the same eigenvalue (i.e the blade is an eigenspace of R_{ab}) Similar remarks apply in the non-simple case to each blade (but with, in general, different eigenvalues)

5 Local Killing Vector Fields.

So far we have considered Killing vector fields defined <u>globally</u> on M, although their local flows were <u>local</u> isometries.

But physics suggests that an observer will only be able to observe Killing vector fields in his neighbourhood.

So suppose, instead of Killing vector fields defined globally on M, we have, for each $p \in M$, an open neighbourhood U of p and a Lie algebra $K(U)$ of Killing vector fields defined on U. The question is; are these Killing vector fields on U merely restrictions, to U, of global Killing vector fields on M?

The answer depends on the choice of U, the dimensions of the various $K(U)$ and the topology of M.

6. Curvature Symmetry.

Suppose now that X is such that its local flows are curvature tensor preserving. Then X is called a curvature collineation, and

$$\mathcal{L}_X R^a{}_{bcd} = 0 .$$

Denote the Lie algebra of smooth curvature collineations on M by $CC(M)$.

Example

$$ds^2 = -dt^2 + h_{\alpha\beta}\, dx^\alpha dx^\beta$$

$[\alpha, \beta, \gamma = 1, 2, 3 \;,\; t = x^4 \;,\; h_{\alpha\beta} = h_{\alpha\beta}(x^\gamma)]$

Then $f(t)\frac{\partial}{\partial t}$ $(= (0,0,0,f(t))$ is in $CC(M)$ for any smooth f.

Problems!

(i) Differentiability

(ii) Dimension of $CC(M)$

(iii) Orbit structure

(iv) Vanishing on open subsets of M.

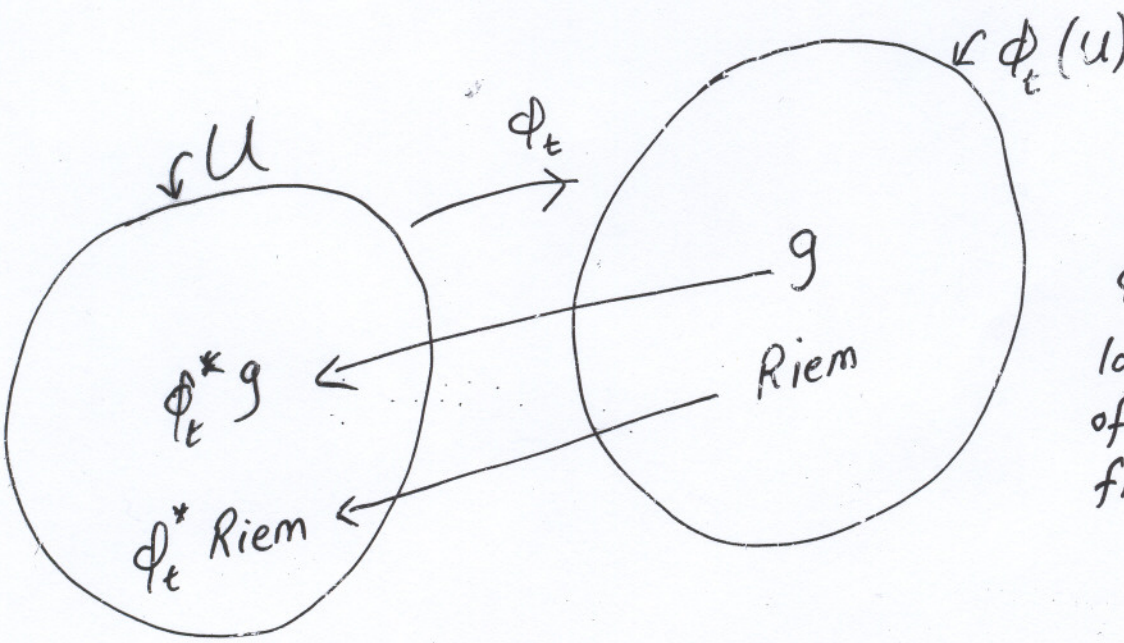
Result "In general" $CC(M) = H(M)$.

Suppose, in some coordinate system, we have metrics g_{ab} and g'_{ab} with the property that their type (1,3) curvature tensors are equal;

$$R'^{a}{}_{bcd} = R^{a}{}_{bcd}.$$

Then, generically, $g'_{ab} = c\, g_{ab}$ where c is constant.

So, generically, we have the following;



ϕ_t is the local flow of a vector field X.

and where Riem is the curvature tensor of g
and ϕ_t^* Riem - - - - - - - - - - - $\phi_t^* g$

So, if X is a <u>curvature collineation</u>, g and $\phi_t^* g$ have same Riem and $\phi_t^* g = c\, g$ (c constant)