What Has String Theory Taught Us About The Quantum Structure of Space-Time

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Based on:

Recent developments in String Theory, to some of which I also have contributed to.

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■ Introduction and Motivations

Gravity is described by

General Relativity (GR)

- is a classical Field Theory with metric of space-time as the field.

- Hence, the "Geometry" of space-time is the dynamical D.O.F.
(Only local properties of the geometry is governed by GR.)

- $S_{E.H.} = \frac{1}{4\pi G_N} \int d^4x \, \sqrt{-g} R.$

Quantum Mechanics (QM)

• Starts with the Noncommutative (NC) Phase Space:

$$[x^i, p_j] = i\hbar\ \delta_i^j$$
$$[x^i, x^j] = 0\ , \qquad [p_i, p_j] = 0$$

• Special Relativity + Q.M. → Quantum Field Theory (QFT).

• Q: Quantization of GR?! OR General Relativity + Q.M. ?!

• Quantization of E.H. action as a classical FT fails, due to (perturbative) non-renormalizability of GR.
What is then the theory of Quantum Gravity?

■ STRING THEORY

Starts with a simple elegant idea:
Everything including the space-time itself is made out of "fundamental strings" or everything including the geometry itself is a state in the Hilbert space of string theory.

- As a model for Q.Gr. string theory should tell us

What is the Quantum Space-Time emerging from string theory?
OR
How does the space-time probed by strings (or other objects available in string theory) look like?

Obviously we have not understood Q.Gr. without knowing the answer to this question.
Putting it another way (in my opinion)

Q.Gr. should be formulated on a Quantum Space-Time

Other Motivations:

• Setting the framework to answer
the Cosmological Constant Problem,
which I believe is the key question in the Quantum Gravity.

• Brings new concepts into physics, e.g. IR/UV connection and non-decoupling of IR and UV degrees of freedom.

• May signal new physics at scales much lower than Planck, as low as 1-10 TeV
and have consequences for various stages of cosmology.

And many more which could be added to the above list.........

Outline of the rest of my talk

■ A brief review on D-branes, as key objects in string theory which be used as probes of space-time.

■ The BFSS Matrix Model, As the first place Quantum Noncommutative stricture of space-time emerged within string theory.

■ The AdS/CFT conjecture, as the best understood example of the Quantum Gravity & as the most rigorous example of formulation of the holographic principle.

■ The conjunction of the two:
The Tiny Graviton Matrix Theory (TGMT)
■ NC structure of space-time, the quantum $AdS_5 \times S^5$ space emerging from the TGMT.
■ Tests and comparisons with the four dim. dual gauge theory picture...

■ Summary & works in progress

D-branes: The key objects in the recent developments of string theory.

- Are defined as DYNAMICAL objects where open strings can end with Dirichlet Boundary Conditions on the directions transverse to the brane.
- They can come in various dimensions, D_p-brane is an object with $p+1$ dimensional worldvolume.
- They preserve 1/2 of the SUSY of the background, i.e. 16 SUSY's in flat space.
- The Low Energy Effective Theory (LEET) which lives on a single D_p-brane is a Supersymmetric $(p+1)$ dim. $U(1)$ gauge theory. In the lowest order that is a SYM.
- For N coincident D_p-branes the LEET becomes a $U(N)$ Supersymmetric Yang-Mills.
- In the field content of a $(p+1)$ dim. SUSY gauge theory, which besides the gauge fields we have $9-p$ scalars and the same number of fermions (gauginos). They are all in the adjoint representation of the group. For $U(N)$, that is $N \times N$ matrices.
- These $9-p$ scalars have a geometric interpretation: they correspond to the fluctuations of the D_p-brane in the $9-p$ directions transverse to the brane.

• As such, for N coincident D_p-branes they become $N \times N$ matrices. That is,

The space-time transverse to D_p-branes is described by $N \times N$ matrices (rather than a simple number).

• In particular we can have D0-branes (D-particles). The theory describing dynamics of N D0-branes is a $(0+1)$ dim. $U(N)$ SYM.

• As a side remark: $D0$-branes are gravitons (gravity waves) of the 11dim. M-theory.
NOTE: 10dim. superstring theories are coming as limits of the 11dim. M-theory via compactifications.

▶ The above observations led Banks-Fischler-Shenker-Susskind (BFSS) to their famous Matrix Theory conjecture:

▸The BFSS Conjecture◂

The Discrete Light-Cone Quantization (DLCQ) of M-theory in the sector with N units of the light-cone momentum is described by dynamics of N $D0$-branes, i.e. a $U(N)$ $0+1$ dim. SUSY Yang-Mills gauge theory:

$$S = \frac{1}{R_-}\int dt\ Tr\Big(D_0 X^i D_0 X^i + \frac{1}{4}[X^i, X^j]^2 + \bar{\Psi} D_0 \Psi + \bar{\Psi}\gamma^i [X^i, \Psi]\Big)$$

• Outcome: Quantized M/String theory leads us to a Quantum space-time described by $N \times N$ matrices.

• The classical (continuum) space-time picture emerges in the

$$R_-,\ N \to \infty, \quad p^+ = \frac{N}{R_-} = fixed.$$

• In this limit the diagonal elements of the matrices gives the "position" of the D0-branes in the 9 dim. space and the off-diagonals are fields on this 9 dim. space.

• The x^+ is the light-cone time and x^- direction is "an emergent" direction.

▸The AdS/CFT◂

The strong statement:

Quantum Gravity (i.e. string theory, the type IIB string theory) on the $AdS_5 \times S^5$ geometry is dual, or equivalent, to a $D = 4$, $\mathcal{N} = 4$ SYM theory, which is a 4dim. conformal field theory (and hence the name CFT), a non-gravitating field theory which could be quantized by ordinary path integral.

- In this duality

$$R^4_{AdS} = l_p^4 N$$
$$l_p^4 = l_s^4 g_s \ , \quad g_s = g^2_{YM}$$

- $N = \#$ the fiveform flux through the S^5

or

Rank of the dual gauge group $U(N)$.

• According to AdS/CFT any state and process in the Gravity side has a counterpart in the gauge theory side and vice-versa, e.g.

Physical (string) states $\longleftrightarrow$ gauge inv. opts. in SYM

String Scattering Amplitudes $\longleftrightarrow$ n-point functions

1/2 BPS D-brane type objects $\longleftrightarrow$ Some specific opts.

There are some comments in order

• Metric of $AdS_5 \times S^5$ (in the global coordinates) is

$$ds^2 = R^2_{AdS}\Big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2\Big) + R_S^2\Big(\cos^2\theta d\phi^2 + d\theta^2 + \sin^2\theta d\tilde{\Omega}_3^2\Big)$$

$R_{adS} = R_S$.

• The AdS/CFT has had many extensions and variations

to CFT's in various dimensions

to less SUSic cases and even

to cases which are not conformal and

.....

• AdS/CFT is a Strong-Weak duality: effective coupling of the gauge theory is the 't Hooft coupling λ

$$\lambda = g_{YM}^2 N = \left(\frac{R_{AdS}}{l_s}\right)^4$$

The effective string coupling is $\frac{1}{N^2}$. Perturbative, weakly coupled string theory $\rightarrow$ large N.

The gravity approx. to string theory can only be trusted when

$$\left(\frac{R_{AdS}}{l_s}\right)^4 \gg 1, \quad i.e. \quad \lambda \gg 1,$$

when the dual gauge theory is not perturbative.

• AdS_5 has a causal boundary $R \times S^3$.

Causal boundary is not formally a part of the space-time but one can send and receive light signals to, in a finite coordinate time.

Boundary is a "covariant notion". That is, it is the same for all observers.

Q: What can the $\mathcal{N} = 4$ gauge theory teach us about the nature of Quantum $AdS_5 \times S^5$?!

In general, it is a hard question to answer and we have only been able to find an answer in a very specific case, if we restrict ourselves to a sector of Opt's in the gauge theory which has a simple dynamics.
Due to SUSY, we have various BPS Opt's which are protected from Quantum Correction and close onto themselves. The more SUSY, the more restricted set of Opt's and their dynamics....
The 1/2 BPS sector:
$\mathcal{N} = 4$ $U(N)$ field content:

$$\begin{gathered} 1 \text{ gauge field } A_\mu, \\ 6 \text{ real scalar fields } \phi_i,\ i = 1, 2, \cdots 6. \\ 4 \text{ Weyl fermions } \psi^I_\alpha,\ I = 1, 2, 3, 4, \alpha = 1, 2. \end{gathered}$$

Take $\phi_5 + i\phi_6 \equiv Z$, the 1/2 BPS Opt's (technically called *Chiral Primary OPt's* are those which are only made out of Z and its powers TrZ^J, $:TrZ^{J_1}TrZ^{J-J_1}:$ or in the most general form:

$$:\prod_{i=1}^{K} TrZ^{J_i}: \quad (\sum_{i=1}^{K} J_i = J).$$

• All 1/2 BPS Opt's with J# Z-fields (# Z-fields is a conserved charge, the R-charge) have the same scaling dimension Δ which is exact and equal to their R-charge J.
• In the sector with a given R-charge J, one can classify all the chiral primaries. It is sol'n to the simple well-posed math problem of

partition of J into any number of non-negative integers, i.e.
Finding set of $\{J_i\} \in \mathbb{Z}$, $\sum_{i=1}^{K} J_i = J$.

Solutions to the above can be given by Young Tableaux of J boxes, which in turn is equivalent to all irreps of $U(J)$.

▶ There are two key questions:
(i) What is the dynamics of the chiral primaries in the $\mathcal{N} = 4$ SYM?
(ii) What is the corresponding gravity sol'ns?

• Gauge Theory Picture:
The action of $U(N)$ $\mathcal{N} = 4$ gauge theory on $R \times S^3$ is

$$S = \frac{1}{4\pi g_{YM}^2} \int dt d\Omega_3 \; Tr \Big[F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi_i)^\dagger (D_\mu \Phi_i) + [\Phi_i, \Phi_j]^2 - \frac{1}{6} \Phi_i^\dagger \Phi_i + fermions \Big]$$

Assuming no dependence on S^3 and that only $\Phi_5 + i\Phi_6 = Z$ (and not $Z^\dagger$) is turned on:

$$S|_{1/2\ BPS} = \int dt Tr\Big[(D_0 Z)^\dagger (D_0 Z) - Z^\dagger Z\Big]$$

which is nothing but simple $N \times N$ harmonic oscillators. BUT, we still have the gauge symmetry and not all the N^2 harm. Oscil. are independent.

One can use the gauge symmetry to diagonalize Z, $Z = diag(z_1, z_2, \cdots, z_N)$ but then

$$\mathcal{Z}_{1/2\ BPS} = \int \frac{DZ^\dagger DZ}{Vol(U(N))}\ e^{S_{1/2BPS}}$$
$$= \int \prod_{i=1}^{N} Dz_i\ e^{S}_{diag} \times \prod_{i>j}(z_i - z_j)$$

The $\prod_{i>j}(z_i - z_j)$ factor is the Van der Mond determinant (Jacobian of transformations).

The 1/2 BPS condition is then translated into

$$\Pi_Z = iZ^\dagger, \quad \Pi_{Z^\dagger} = -iZ$$

Implying that

$$[Z, Z^\dagger] = 1$$

in the BPS sector.

• The above system is equivalent to the system of N free fermions in one dimension.

• This system is also equivalent to a system of N $2d$ fermions in the external magnetic field. The BPS condition tells us that the fermions are sitting in the Lowest Landau Level (LLL).

As the original $\mathcal{N} = 4$ gauge theory we start with is a conformal theory one can always, by a conformal scaling, make the strength of the magnetic field equal to the density of the particles.

• As a Quantum Hall System, the 1/2 BPS sector has a filling factor=1. That is, it has a manifest particle/quasihole exchange symmetry.

- Gravity Picture:

Lin-Lunin-Maldacena (LLM) [hep-th/0409174] constructed all the sol'ns to SUGRA equations with exactly the symmetry of the 1/2 BPS chiral primary Opt's.

The LLM solutions are geometries with $SO(4)\times SO(4)$ isometries and have a globally defined time-like (or light-like) Killing directions:

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx_i^2) + ye^{G} d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2$$

$$h^{-2} = 2y\cosh G\ , \quad z = \frac{1}{2}\tanh G, \qquad y \in [0, \infty)$$

$$\epsilon_{ij}\partial_i V_j = \frac{1}{y}\partial_y z\ ; \quad \epsilon_{ij}\partial_j z = y\partial_y V_i$$

The above two equations imply that $\frac{1}{y^2}z \equiv \Phi$ satisfies a six dim. Laplace equation:

$$\partial_i^2\Phi + \frac{1}{y^3}\partial_y(y^3\partial_y\Phi) = 0$$

OR

$$\boxed{\partial_i^2 z + y\partial_y(\frac{1}{y}\partial_y z) = 0}$$

- Smoothness condition then forces $z(x_i; y)$ to take values

$$z(x_i; y = 0) = \pm\frac{1}{2}$$

▶ Comments:

- All the LLM sol'ns are then given by

$$z = \frac{y^2}{\pi} \int d^2 x' z_0(x_i; 0) \frac{1}{\left(y^2 + (x_i - x_i')^2\right)^2}$$

As $z_0(x_i; 0)$ only takes values $\pm 1/2$ one may use a Black/White color-coding on x_i plane to distinguish regions with $z_0 = +1/2$ from the regions with $z_0 = -1/2$.

- Quantization of the fiveform flux implies:

Area of the Black region (in 10 dim. Planck units) should be quantized. This in turn implies that

$$[x_1, x_2] = i l_p^4$$

or if we introduce $Z = \frac{1}{\sqrt{2}}(x_1 + i x_2)$,

$$[Z, Z^\dagger] = i l_p^4 \ .$$

That is, both (semi-classical) gravity and gauge theory lead us to the same result:

The (x_1, x_2) plane in the LLM geometry is a NonCommutative Moyal plane.

But this is not a satisfactory uniform picture, what about the rest of the directions?!....

The Tiny Graviton Matrix Theory

The TGMT conjecture:

DLCQ of type IIB string theory on the $AdS_5 \times S^5$ or the ten dim. max. SUSic plane-wave background in the sector with J units of the the light-cone momentum is described by a $U(J)$ $0+1$ dim. supersymmetric gauge theory; i.e. a $U(J)$ SUSY QM, with the Hamiltonian

$$\mathcal{H} = R_- \; Tr \Bigg[\frac{1}{2}\Pi_I^2 + \frac{1}{2}\left(\frac{\mu}{R_-}\right)^2 X_I^2 + \frac{1}{2 \cdot 3! g_s}[X^I, X^J, X^K, \mathcal{L}_5][X^I, X^J, X^K, \mathcal{L}_5] - \frac{\mu}{3! R_- g_s}\Big(\epsilon_{ijkl} X^i [X^j, X^k, X^l, \mathcal{L}_5] + \epsilon_{abcd} X^a [X^b, X^c, X^d, \mathcal{L}_5] \Big) + Fermions \Bigg]$$

where $I, J, K = 1, 2 \cdots, 8$, $I = \{i, a\}$ and $i, j = 1, 2, 3, 4$ and $a, b, c = 5, 6, 7, 8$.

• $\mathcal{L}_5$ is a $J \times J$ unitary matrix where

$$\mathcal{L}_5^2 = \mathbf{1}_{J\times J}\ , \quad Tr\mathcal{L}_5 = 0$$

• $$[F_1, F_2, F_3, F_4] \equiv \epsilon^{ijkl} F_i F_j F_k F_l$$

is the quantized Nambu 4-bracket. Nambu brackets are a direct generalization of Poisson brackets.
It defines an algebra, not a symplectic one though.
This algebra is closely related to the Quantum version of the Area Preserving Diffeomorphisms of a four (or three) dimensional surface.

• The TGMT Hamiltonian is obtained from the Discretized or Quantized version of the action of a D3-brane in the ten dimensional plane-wave background.

• The TGMT enjoys the following $U(J)$ gauge symmetry:

$$X^I, \Pi^I \to U X^I, \Pi^I U^{-1}\ , \quad U \in U(J)$$
$$\mathcal{L}_5 \to U \mathcal{L}_5 U^{-1}$$

• Although a gauge theory, TGMT is not a Yang-Mills theory.

• The $U(J)$ gauge theory of the TGMT is the Quantized (or discretized) version of the Area Preserving Diffeomorphisms of the threebrane.

• The Physical states of the TGMT are subject to the Gauss Law constraint:

$$\left(i[X^I, \Pi^I] + 2\Psi^{\dagger\alpha\beta}\Psi_{\alpha\beta} + 2\Psi^{\dagger\dot{\alpha}\dot{\beta}}\Psi_{\dot{\alpha}\dot{\beta}}\right)|\phi\rangle_{phys} = 0$$

• TGMT is the theory of J tiny gravitons, the cousins of the D0-branes. In this sense the TGMT parallels the BFSS.

Conceptually what TGMT shares with BFSS is that in both cases this is the gravitons or gravity waves or metric fluctuations which are used to formulate a quantum gravity.

What is then a tiny graviton?!

(A detour to the Definition of Tiny Gravitons

Consider a Spherical D3-brane in the $AdS_5 \times S^5$ background which moves on an $S^1 \in S^5$ with angular momentum J and follows a light-like geodesic. This brane, which is called a Giant Graviton is STABLE, 1/2 BPS and has radius:

$$R_{giant}^2 = R_{AdS}^2 \times \frac{J}{N}$$

IF J takes its minimum value, $J = 1$, we have

$$R^2 = R_{AdS}^2 \times \frac{1}{N} = l_p^2 \times \frac{1}{\sqrt{N}} \equiv l^2$$

As we see in the large N, $l \ll l_p$, and hence the name TINY GRAVITON is an appropriate one.

In short, Tiny Graviton is a spherical three-brane of very small (sub-Planckian) *classical* size.
It is on one hand a graviton and on the other hand a D-brane.

In the Language of the TGMT Hamiltonian

$$l^2 = \frac{\mu}{R_-} l_s^2 g_s$$

end of the detour)

1/2 BPS configurations of the TGMT are all the matrices which render

$$H = 0$$

that is,

$$\Pi_I = 0, \quad X^a = 0$$
$$[X_i, X_j, X_k, \mathcal{L}_5] = -l^2 \epsilon_{ijkl} X_l$$

The sol'ns to the above equations are Non-commutative, Fuzzy three sphere S_F^3. To completely define S_F^3 beside the above four-bracket equation we need the radius:

$$R^2_{S^3_F} \equiv \sum_{i=1}^{4} X_i^2 = l^2 J$$

l is the fuzziness and in the $l \to 0$, $R_{S^3_F} = fixed$ we recover the round three sphere.
NOTE that the radius squared of the fuzzy three sphere is quantized in units of l^2.

The above mentioned *single sphere* solution is only one of the solutions to the four-bracket equation. In general we can have fuzzy sphere solutions which are of the form of concentric fuzzy three spheres and the sum of their radii squared (in units of l^2) is J.
For more details see [hep-th/0501001].

■ Relation to LLM geometries

What we learn from the TGMT about the LLM geometries is then:

• The two three spheres in the LLM geometries are classical versions of a fuzzy three sphere, whose radii are quantized.

• The quantization of the three sphere radii implies that the (x_1, x_2) plane in the LLM geometries is a NC Moyal plane.

NOTE: In the class of the LLM geometries corresponding to the TGMT sol'ns the x_1 appears as the light-like circle and (x_1, x_2) plane is indeed a cylinder, a NC, fuzzy cylinder, *i.e.*

$$[x_1, x_2] = il^2 \times \left(\frac{R_- \mu^2}{l_s}\right) = il_p^4$$

• The radius quantization also implies the discreteness of the spectrum of y coordinate.

And Hence all the eight transverse directions in the quantized LLM geometries are quantized as explained above.

SUMMARY & OUTLOOK

• A picture of Quantum Space-Time should emerge from theory of Quantum Gravity and in particular String theory.

One may even start with a specific Quantum Space-Time and try to build a Q.Gr. based on that. This may turn out to be string theory or otherwise?!

• Within string theory we have some examples in which we have encountered NC (Quantum) structure in space-time. Here I reviewed some cases which involves gravity.

• The NC structure uncovered depends on the probe and also the background. However, the point is that in each case it is possible to find an appropriate probe. Besides the conceptual point mentioned above, is there anything fundamentally in common in different case?! This is yet to be explored.

• The NC structure is correlated with the fluxes in the background.

Things to be studied in further detail:

- Generically the NC space-times show IR/UV mixing. This may have implications for the Cosmological Constant problem........

- The above ideas could be used to find a "HOLOGRAPHIC" formulation for gravity on various backgrounds and may also be used to resolve issues in counting micro-states of a black-hole.........