Computation of Equilibrium & Micro-instabilities in tokamak and Stellarator Plasmas

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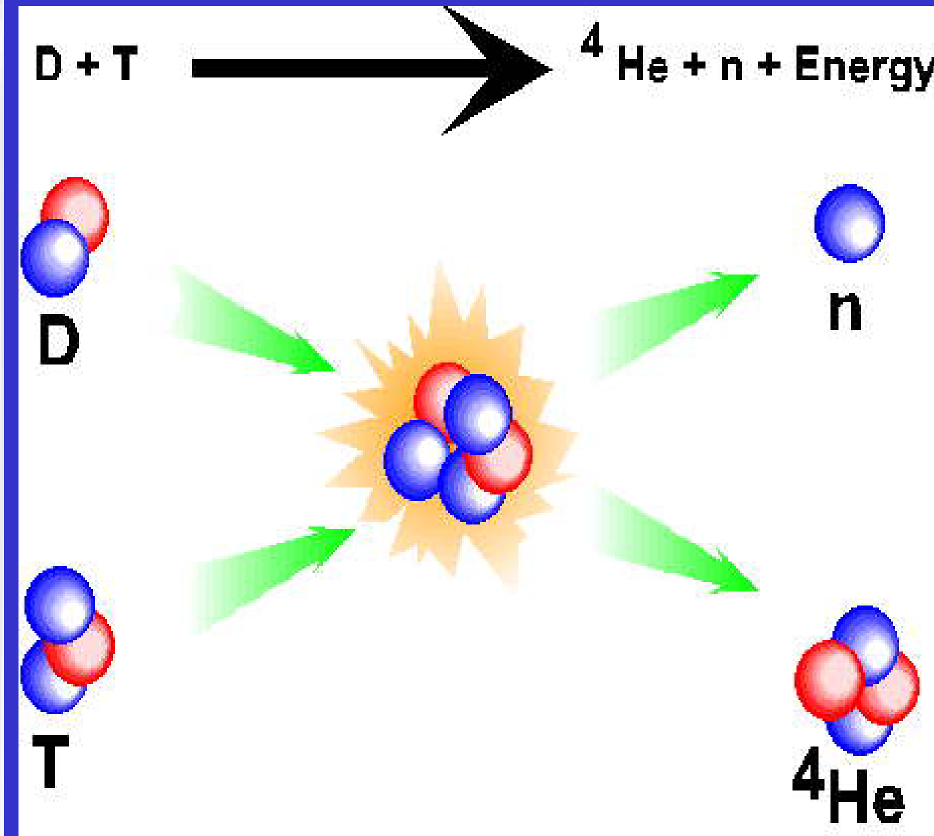
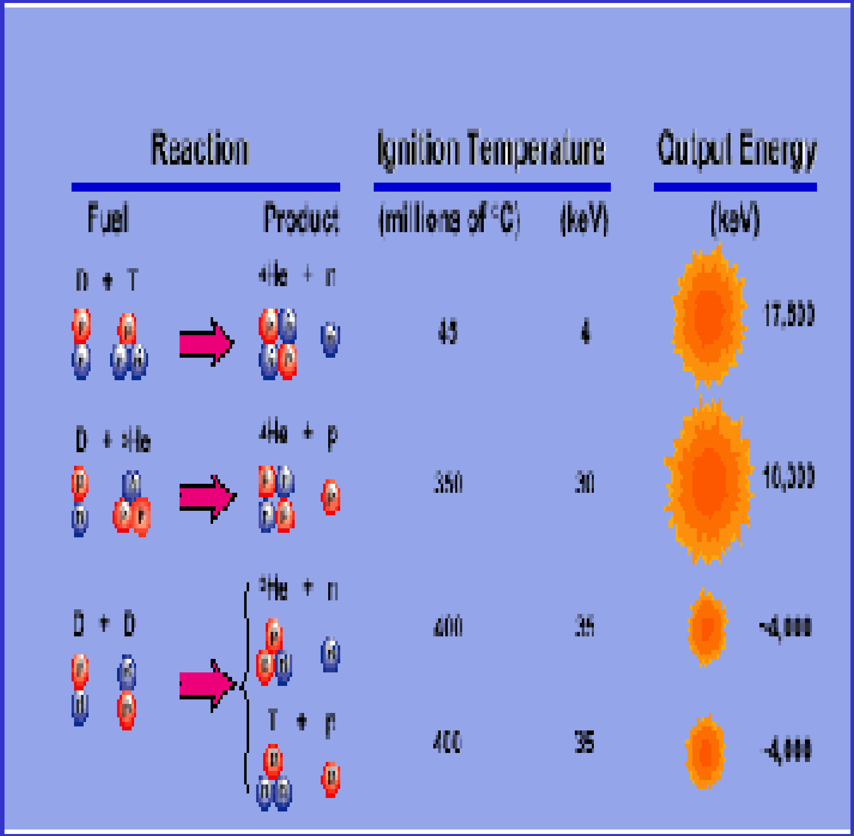
INTRODUCTION

Fusion



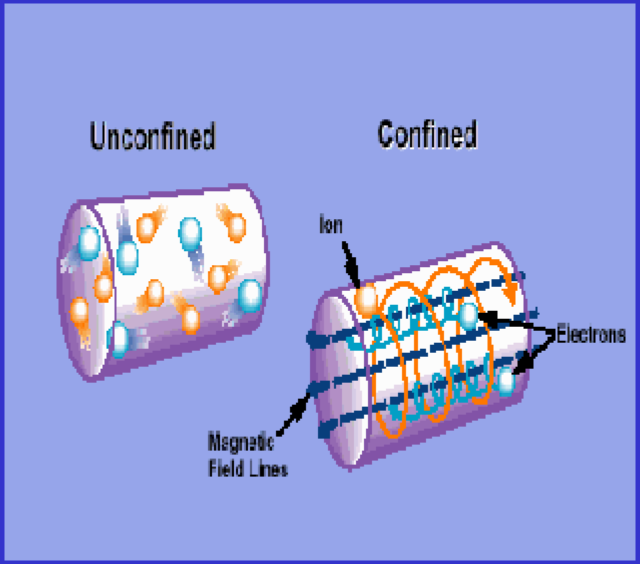
D + T → ^{4}He + n + Energy
D
T
n
^{4}He



Reaction
Fuel
Product
Ignition Temperature
(millions of °C)
(keV)
Output Energy
(keV)
D + T → ^{4}He + n
45
4
17,600
D + ^{3}He → ^{4}He + p
350
30
18,300
D + D → ^{3}He + n
400
35
~4,000
T + p
400
35
~4,000

Magnetic Confinement

tokamak

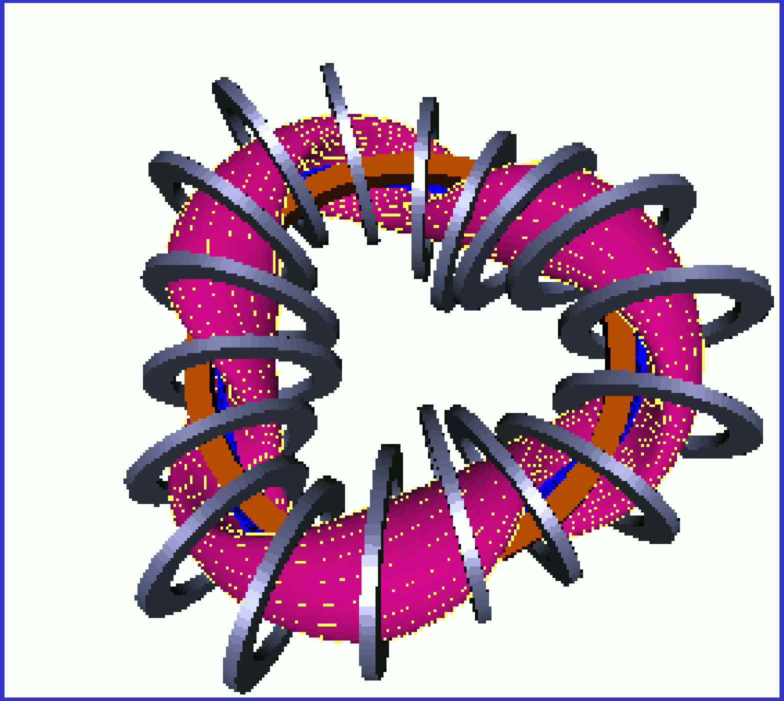


Lawson criteria

$n\tau > 10^{-19}$ s m^{-3}

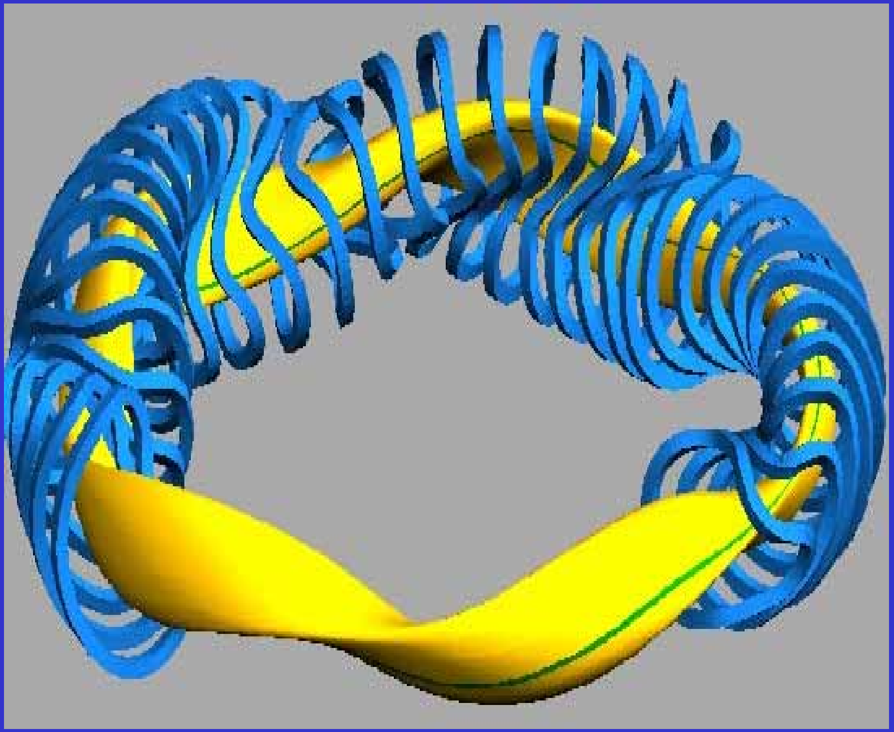
Stellarators

H1-NF



3 Field period

Wendelstein 7-X



5 Field period

Layout of the talk

- Introduction to numerical scheme
- Magnetic field in 3D geometries
- Calculations of equilibrium quantities

  mod B, curvature, LMS, ILMS
- Drift wave model and reduction to eigen value problem

  ¡ delta, ITG, TEM
- Boundary conditions and numerical method
- A few results

Numerical Scheme

VMEC Code

(Generalized Cylindrical Coordinates)

Mapper Code

(Boozer Flux Coordinates)

Drift Code

The VMEC Code

Covariant and contravariant vectors

The position vector :

$$\mathbf{r}_p = (R\cos\phi_c, R\sin\phi_c, z).$$

Cylindrical coordinates :

$$\left.\begin{aligned} R &= \sum_{m=0}^{n_p}\sum_{n=-n_t}^{n_t} R_{mn}(s)\cos(m\theta + nN\zeta) \\ \phi_c &= \zeta - \frac{2\pi}{N}\sum_{m=0}^{n_p}\sum_{n=-n_t}^{n_t}\phi_{mn}(s)\sin(m\theta + nN\zeta) \\ z &= \sum_{m=0}^{n_p}\sum_{n=-n_t}^{n_t} z_{mn}(s)\sin(m\theta + nN\zeta). \end{aligned}\right\}$$

The covariant basis vectors :

$$\mathbf{e}_s = \frac{\partial \mathbf{r}_p}{\partial s}, \qquad \mathbf{e}_\theta = \frac{\partial \mathbf{r}_p}{\partial \theta}, \qquad \mathbf{e}_\zeta = \frac{\partial \mathbf{r}_p}{\partial \zeta},$$

The contravariant basis vectors :

$$\nabla s = \frac{\mathbf{e}_\theta \times \mathbf{e}_\zeta}{J}, \qquad \nabla\theta = \frac{\mathbf{e}_\zeta \times \mathbf{e}_s}{J}, \qquad \nabla\zeta = \frac{\mathbf{e}_s \times \mathbf{e}_\theta}{J}.$$

The jacobean for Boozer coor. :

$$J \equiv \mathbf{e}_s \cdot \mathbf{e}_\theta \times \mathbf{e}_\zeta = \frac{\dot{\psi}}{B^2}(B_\theta + qB_\zeta) \ ,$$

Magnetic field configuration

- The magnetic field in Boozer coordinates:

$$\mathbf{B} = \nabla\alpha \times \nabla\psi = \dot{\psi}\nabla\alpha \times \nabla s,$$

$$\dot{\psi} \equiv \frac{d\psi}{ds} = \frac{B_0 \bar{a}^2}{2q}$$

$$\alpha = \zeta - q\theta$$

$$s = 2\pi\psi/\psi_p$$

- The field line curvature : $\boldsymbol{\kappa}(\equiv \mathbf{e}_\parallel \cdot \nabla \mathbf{e}_\parallel)$

$$\boldsymbol{\kappa} = \frac{\kappa_n}{\sqrt{g^{ss}}}\nabla s + \frac{\kappa_g}{\sqrt{g^{ss}}}\left(\frac{\dot{\psi} g^{ss}}{B}\right)(\nabla\alpha - \Lambda\nabla s)$$

- The normal & geodesic curvatures:

$$\kappa_n = \boldsymbol{\kappa} \cdot \frac{\nabla s}{|\nabla s|}, \qquad \kappa_g = \boldsymbol{\kappa} \cdot \left(\frac{\nabla s}{|\nabla s|} \times \mathbf{e}_\parallel\right)$$

- The local magnetic shear:

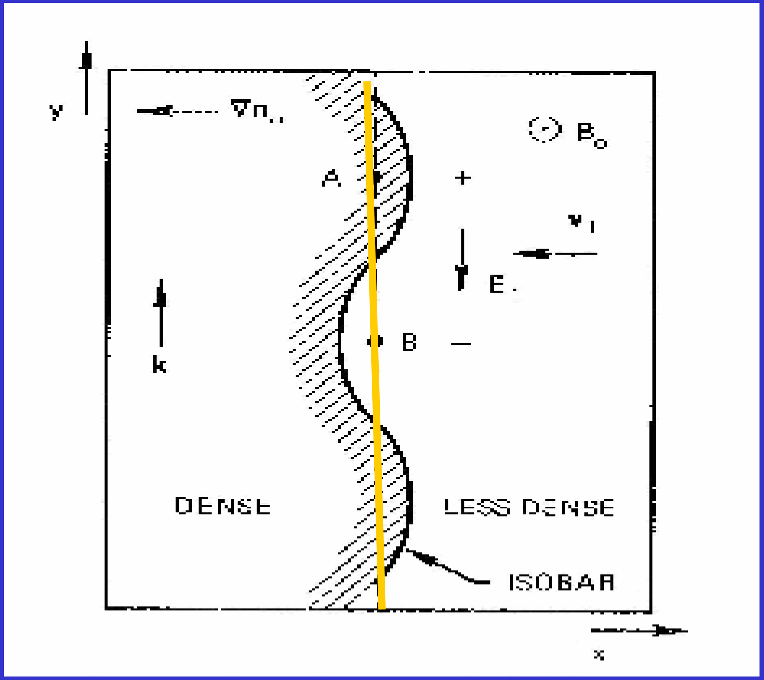
$$S = \left(\frac{\nabla s}{|\nabla s|} \times \mathbf{e}_\parallel\right) \cdot \nabla \times \left(\frac{\nabla s}{|\nabla s|} \times \mathbf{e}_\parallel\right) = \left(\mathbf{e}_\parallel \cdot \nabla\right)\Lambda .$$

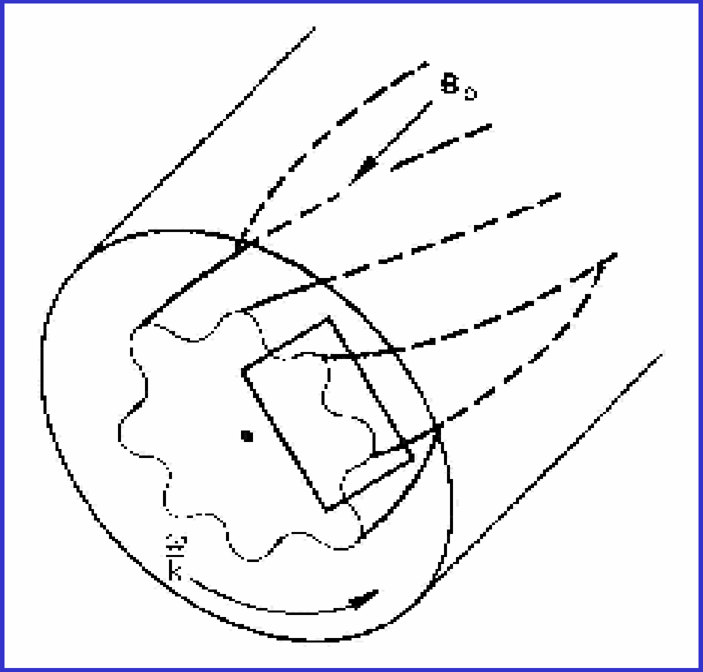
- Integrated LMS

$$S = \frac{\dot{\psi}^2 |\nabla s|^2}{JB^2}\left[-\frac{dq}{ds} + \left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right)\left\{\frac{(\nabla\zeta - q\nabla\theta)\cdot\nabla s}{|\nabla s|^2}\right\}\right].$$

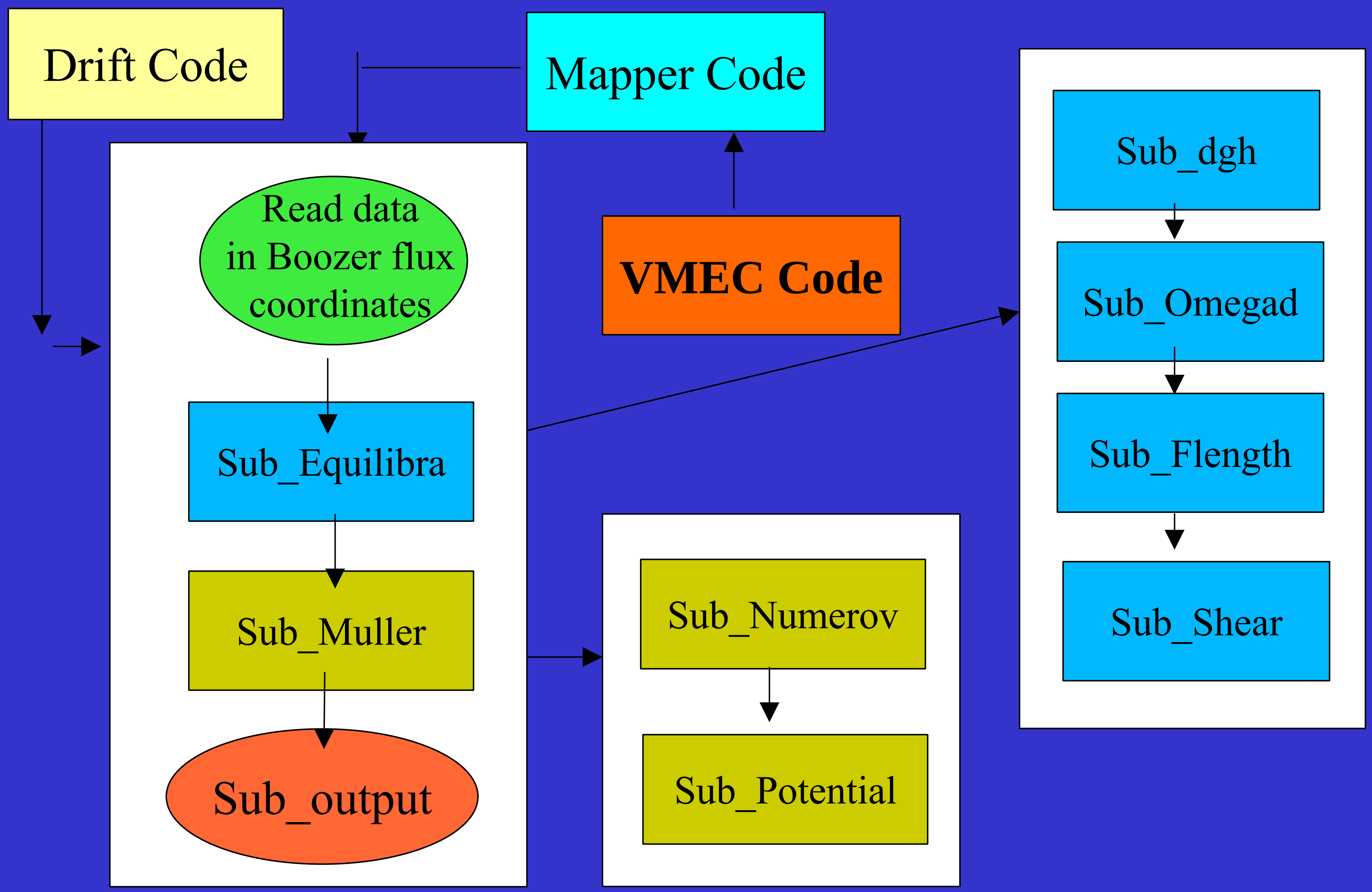
The Drift Code

Simple Picture of Drift Waves





Drift code and numerical scheme



Ballooning formalisim & WKB approx.

$$\hat{\phi}(\mathbf{r}) = \Phi(\psi, \alpha, \zeta) \exp[-i\epsilon^{-1} S(\psi, \alpha, \zeta)]$$

$$\hat{\phi} = \Phi(\zeta) \exp[-i\epsilon^{-1} S(\psi, \alpha)]$$

$$\alpha(\psi, \theta_o, \zeta_o) \to \alpha(\psi, \theta_o + 2\pi, \zeta_o)$$

$$\alpha(\psi, \theta_o, \zeta_o) \to \alpha(\psi, \theta_o, \zeta_o + 2\pi),$$

$$\mathbf{k}_\perp = i\nabla_\perp \ln\hat{\phi} = \epsilon^{-1}\nabla_\perp S = \epsilon^{-1}\left[\frac{\partial S}{\partial \alpha}\nabla\alpha + \frac{\partial S}{\partial \psi}\nabla\psi\right]$$

$$= \epsilon^{-1}\frac{\partial S}{\partial \alpha}\left[\nabla\zeta - q\nabla\theta - \left\{\theta\frac{dq}{d\psi} - \frac{\partial S/\partial\psi}{\partial S/\partial\alpha}\right\}\nabla\psi\right]$$

$$\frac{\partial S/\partial\psi}{\partial S/\partial\alpha} = (\theta_o + \theta_k)\frac{dq}{d\psi}.$$

$$\mathbf{k}_\perp = \epsilon^{-1}\frac{\partial S}{\partial \alpha}\left[\nabla\zeta - q\nabla\theta - \left(\frac{\zeta - \zeta_o}{q} - \theta_k\right)\frac{dq}{d\psi}\nabla\psi\right].$$

$$\zeta - q\theta = \alpha = \zeta_o - q\theta_o$$

Eigen value problem
i-delta model

Drift wave equation:

$$\frac{d^2\Phi}{d\zeta^2} + U(\zeta,\omega) = 0$$

Effective potential using i-delta model:

- ❖ The grad B plus curvature drift
- ❖ The diamagnetic drift
- ❖ FLR effects
- ❖ delta

The parallel gradient operator:

$$\nabla \cdot \nabla_{\|} = \left(\frac{\dot{\psi} q}{JB}\right)^2 \left.\frac{d^2}{d\zeta^2}\right|_{field\ line}.$$

Eigen value problem (i-delta model)

Drift wave equation:

$$\frac{d^2\Phi}{d\zeta^2} + U(\zeta,\Omega)\Phi = 0,$$

Effective potential using i-delta model:

$$U(\zeta,\Omega) = -\frac{1}{F}\left(\frac{JB}{q\bar{R}\dot{\psi}}\right)^2 \left\{(\Omega_* + F\Omega_d)\chi\Omega - (1+b+i\delta)\,\Omega^2\right\};$$

Where

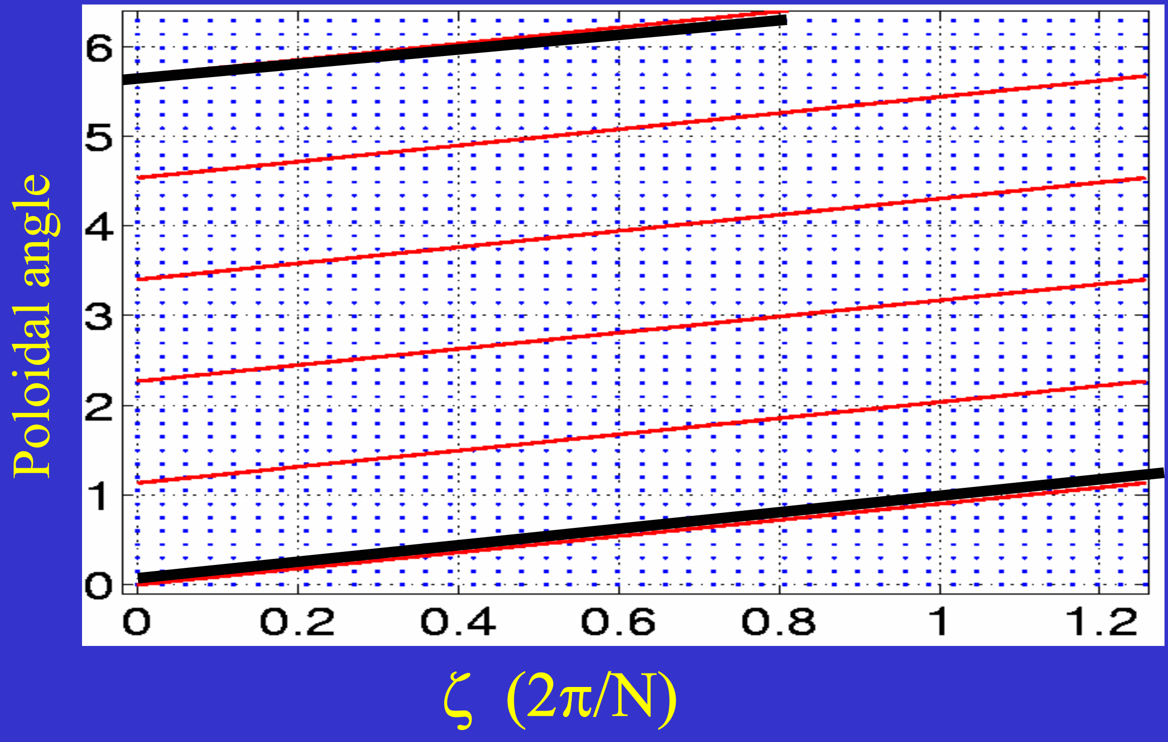
$$F = 1+\tau, \qquad \Omega_* = \Omega_*(s) = \frac{2\bar{R}}{\bar{a}L_n}, \qquad L_n^{-1} = \frac{d\ln n_0}{ds}, \qquad \Omega = \frac{\bar{R}\omega}{c_s},$$

$$\chi = \epsilon^{-1}\frac{q\rho_{s0}}{\bar{a}}\frac{\partial S}{\partial \alpha}, \qquad \tau = \frac{T_i}{T_e}, \qquad c_s = \sqrt{\frac{T_e}{m_i}}, \qquad b = \frac{B_0^2\chi^2|\hat{\mathbf{k}}_\perp|^2}{B^2},$$

$$\Omega_d = \Omega_d(s,\alpha,\zeta) = B_0\bar{R}\left(\frac{\mathbf{B}\times(\boldsymbol{\kappa}+\nabla\ln B)}{B^2}\right)\cdot\hat{\mathbf{k}}_\perp, \qquad \rho_{s0} = \frac{c_s}{eB/m_i}$$

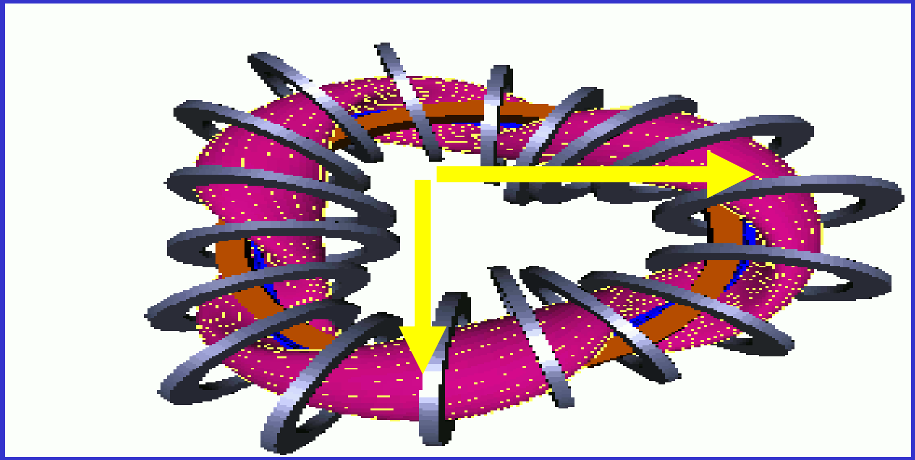
Numerical Method

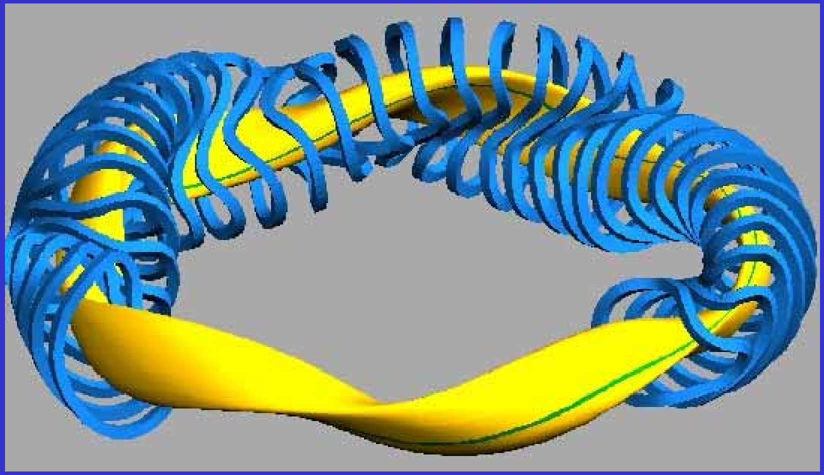
Grid points and field line in one field period of W7-X



Straight line represents the α=0 line

One Field Period of H1-NF



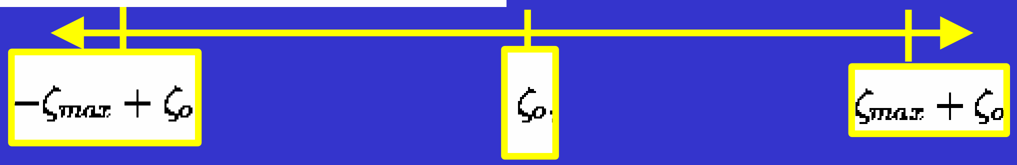


Numerical method

Drift wave equation:

$$\frac{d^2\Phi}{d\zeta^2} + U(\zeta,\Omega)\Phi = 0,$$

The function $U(\zeta,\Omega)$ on the domain $[-\zeta_{max}+\zeta_o, \zeta_{max}+\zeta_o]$ for a given Ω.



The eigen function Φ at $\zeta = \pm\zeta_{max}$ and Φ' are calculated by using sixth order Numerov scheme:

$$\Phi(\zeta-\Delta\zeta)\left[1+\frac{(\Delta\zeta)^2}{12}U(\zeta-\Delta\zeta)\right] = \left[2-\frac{5}{6}(\Delta\zeta)^2U(\zeta)\right]\Phi(\zeta) - \left[1+\frac{(\Delta\zeta)^2}{12}U(\zeta+\Delta\zeta)\right]\Phi(\zeta+\Delta\zeta).$$

The eigen frequency is determined which satisfies the condition:

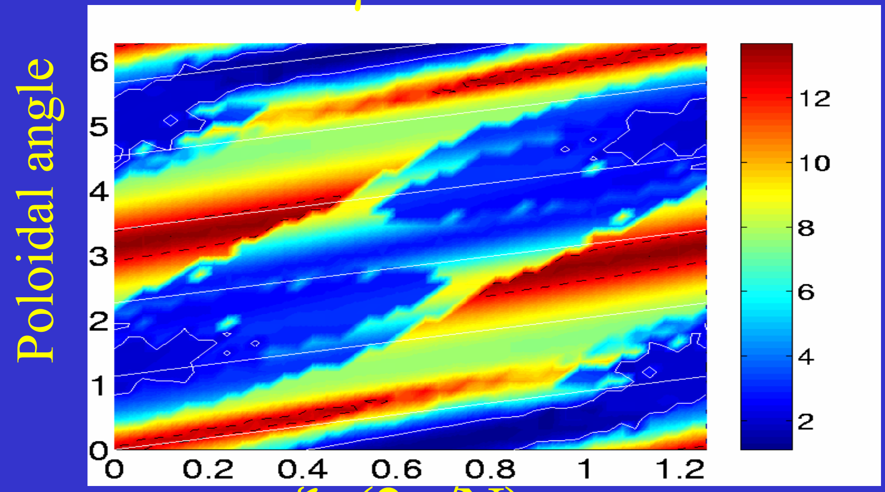
$$f(\Omega) = \frac{\Phi'_+}{\Phi_+} - \frac{\Phi'_-}{\Phi_-} = 0$$

on the complex frequency plane.

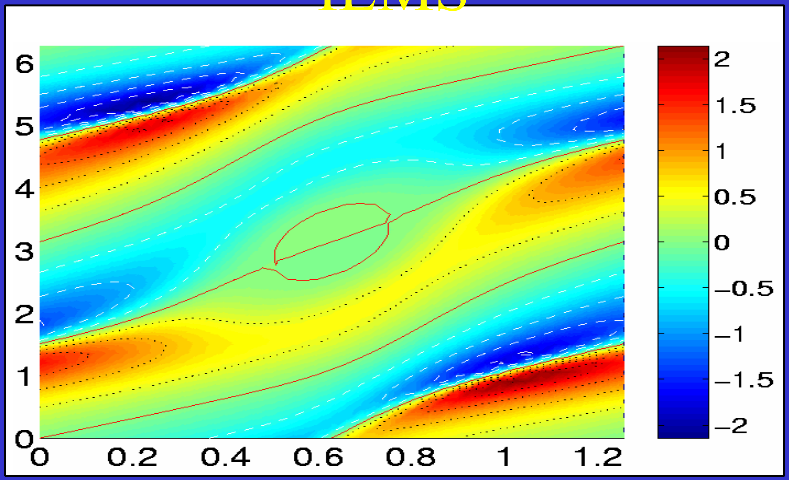
B

Growth rate in one field period (i-delta) W7-X

γ



ILMS



The normalized growth rate $\gamma=(R\gamma/C_s)$ the most unstable modes on the magnetic surface s =3/4, $\theta_\kappa=0.\delta=0.001$, $\varepsilon_n=0.1$

Eigenvalue problem ITG model

Drift wave equation

$$\frac{d^2\Phi}{d\zeta^2}+U(\zeta,\omega)=0$$

- The effective potential includes:
  - The diamagnetic drift
  - The grad B plus curvature drift
  - The temperature gradients
  - The FLR effects
  - Electrons are Boltzmannian

Eigenvalue problem ITG model

- Drift wave equation:

$$\frac{d^2\Phi}{d\zeta^2} + U(\zeta,\Omega)\Phi = 0,$$

Effective potential using ITG model :

$$U = -\left(\frac{2\chi_i JB}{\bar{a}\epsilon_n q\bar{R}\dot{\psi}}\right)^2 \left[(H^{-1} - \frac{\bar{a}\epsilon_n\Omega_d}{2})\Omega - \left\{H^{-1} + \left(\frac{\chi B_0}{B}\right)^2 \left(\hat{\mathbf{k}}_\perp \cdot \hat{\mathbf{k}}_\perp\right)\right\}\Omega^2\right]$$

$$H = 1 + \tau^{-1} + \frac{\tau^{-1}\left[(2/3)\,\Omega + (\eta_i - (2/3))\right]}{\Omega + (5/6\tau)\,\bar{a}\epsilon_n\Omega_d},$$

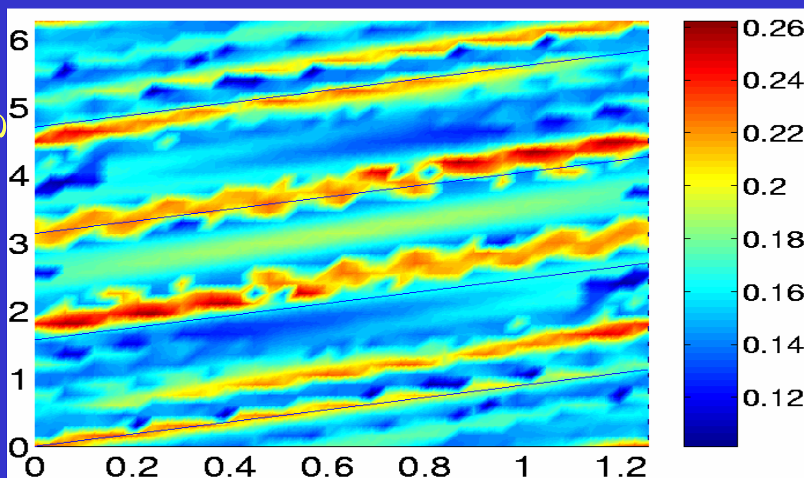
Where

$$\Omega = \omega/\omega_{*e},\ \ \epsilon_n = L_n/\bar{R}\,,\ \chi = (\epsilon\bar{a})^{-1} q\rho_{s0}\partial S/\partial\alpha,$$

Growth rate (ITG) in one field period

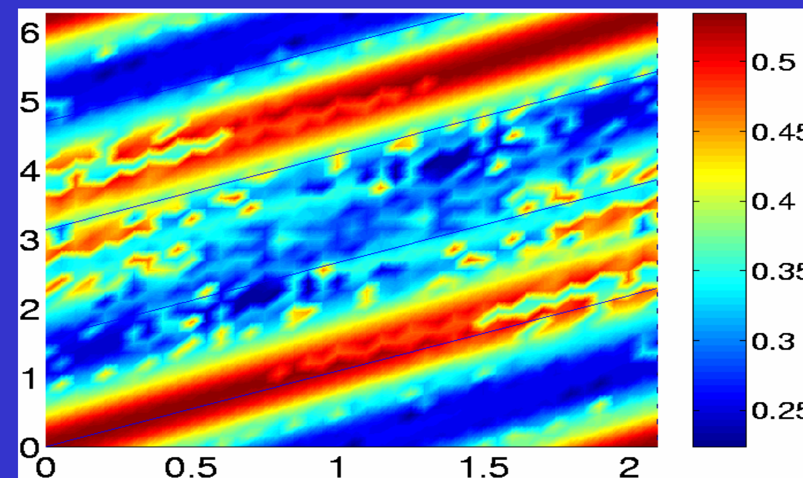
W7-X

Poloidal angle



$\zeta\ (2\pi/N)$

H1-NF



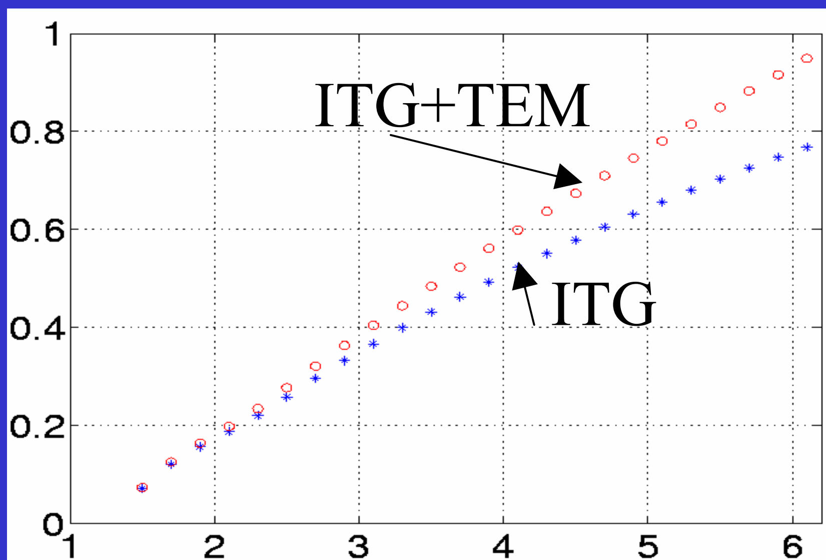
$\zeta\ (2\pi/N)$

The growth rate $\gamma=(\gamma/\omega_{*e})$ of the most unstable modes on the magnetic surface s =3/4, $\theta_\kappa=0$, $\tau=1$, $\varepsilon_n=0.1$ and b=0.1.

Dissipative Trapped Electrons

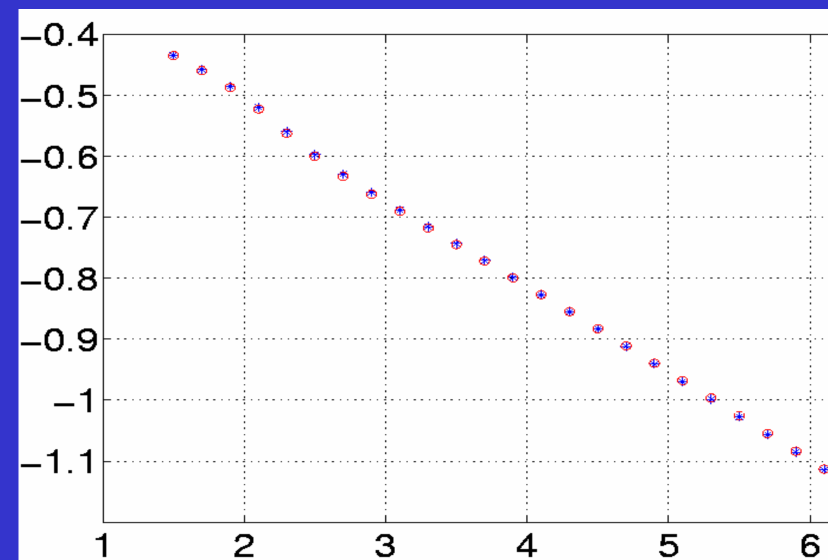
Growth rate (ITG+TEM)
W7-X

γ/ω_{*e}



η_i

Ω/ω_{*e}

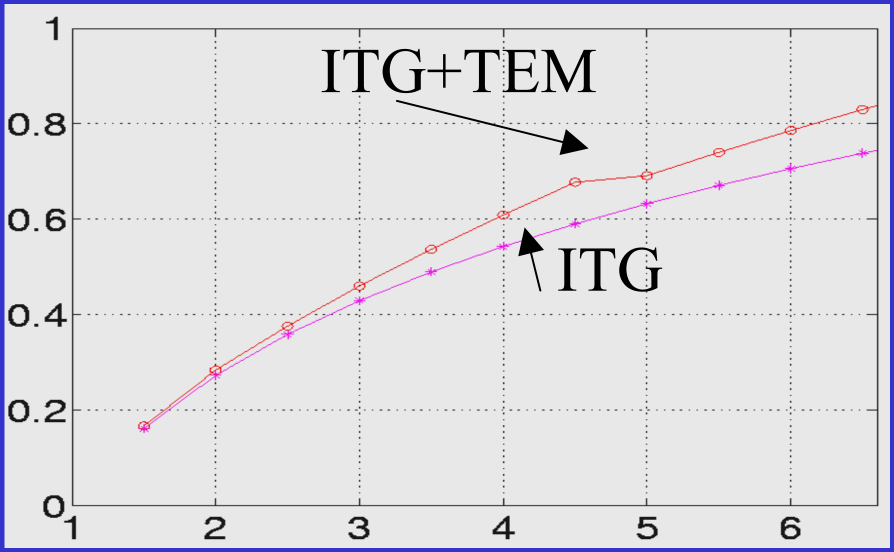


η_i

The growth rate γ=(γ/ω_{*e}) and the eigen frequency Ω(=ω/ω_{*e}) of the most unstable modes VS η_i on the magnetic surface s =0.7, θ_κ=0, τ=1, ε_n=0.1 and b=0.1.

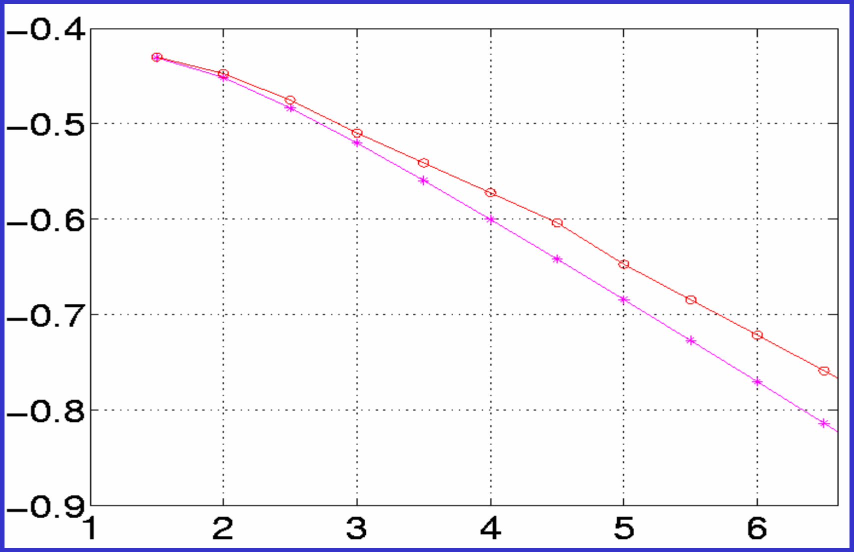
Growth rate (ITG+TEM)
H1-NF

γ/ω_{*e}



η_i

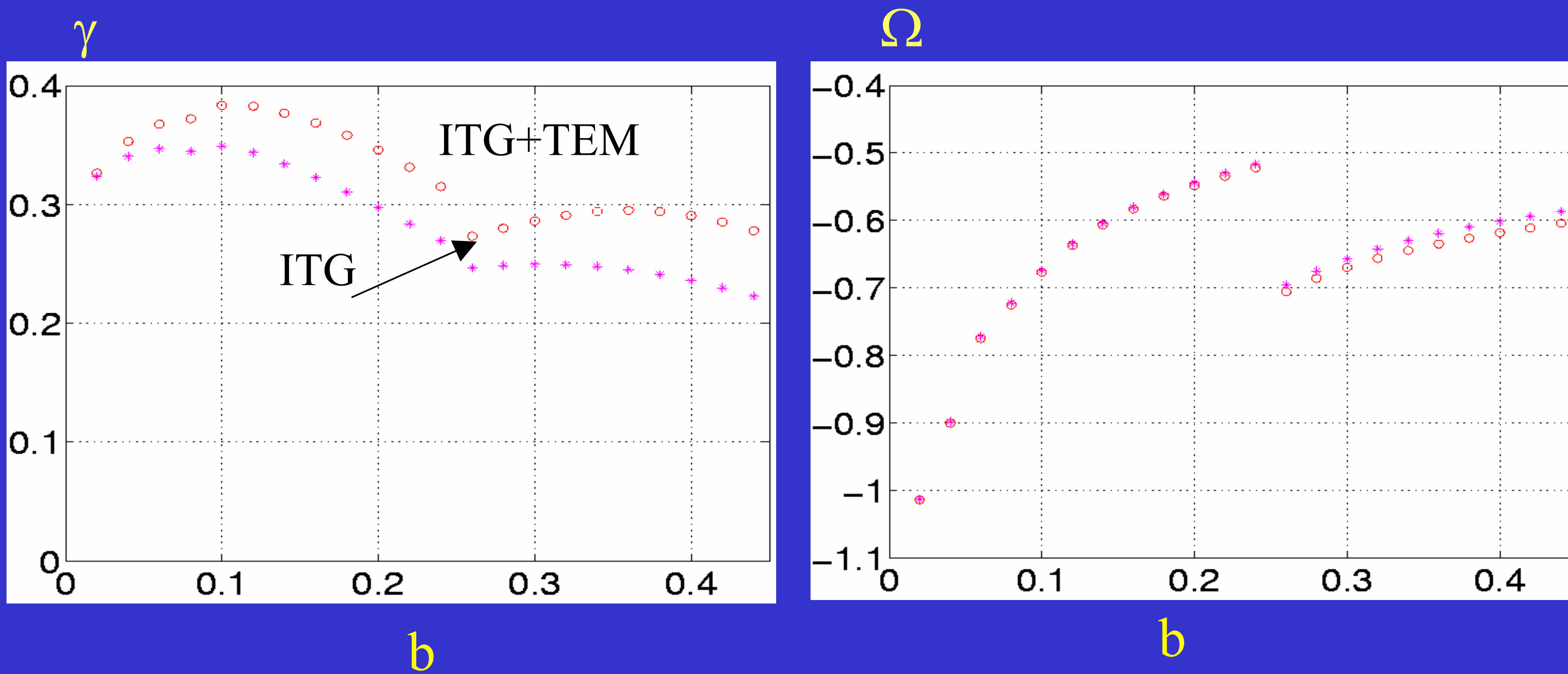
Ω/ω_{*e}



η_i

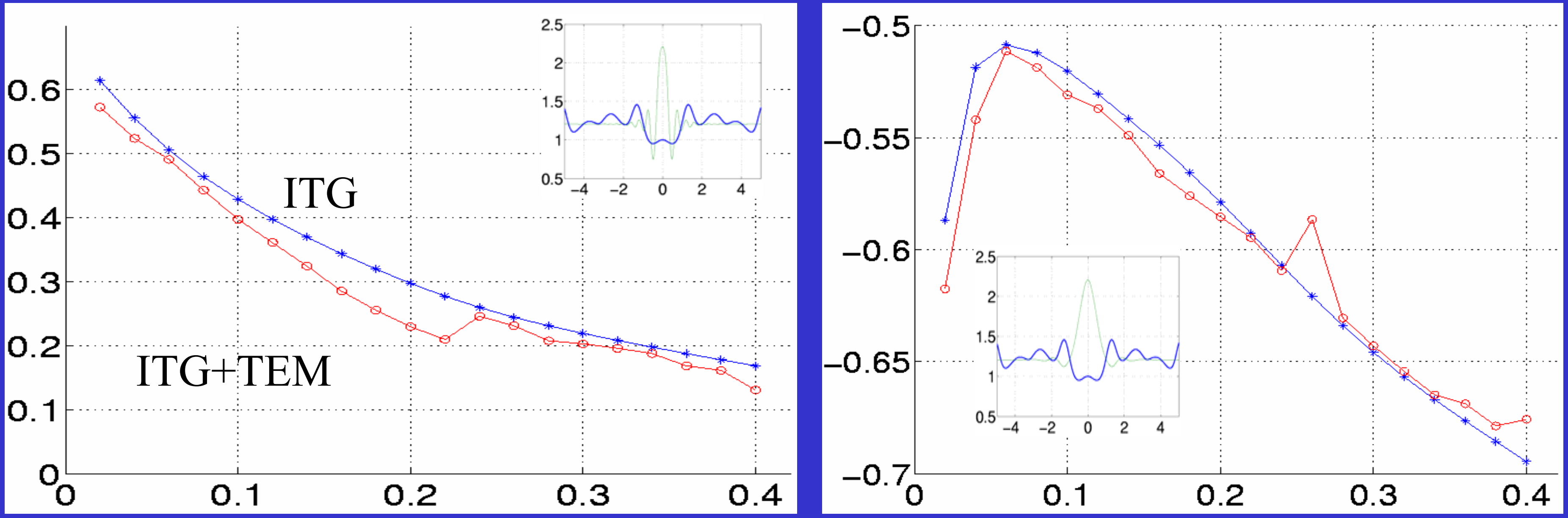
The growth rate γ=(γ/ω_{*e}) and the eigen frequency Ω(=ω/ω_{*e}) of the most unstable modes VS η_i on the magnetic surface s =0.7, θ_κ=0, τ=1, ε_n=0.1 and b=0.1.

Growth rate (ITG+TEM) W7-X



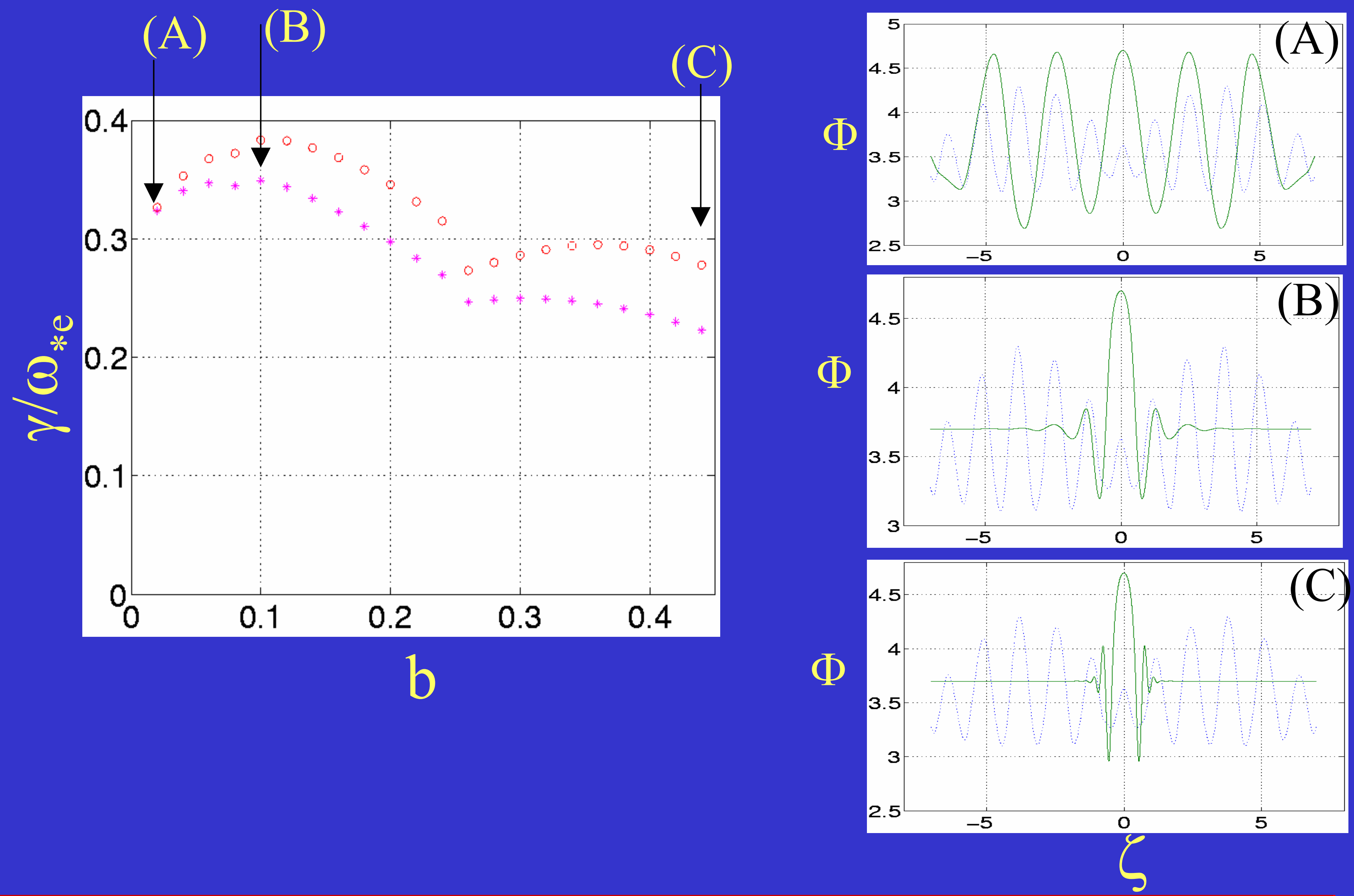
The growth rate γ (= γ/ω_{*e}) and the eigen frequency Ω (=ω/ω_{*e}) of the most unstable modes versus b on the magnetic surface $s = 0.7$, $\theta_\kappa = 0$, $\tau = 1$, $\varepsilon_n = 0.1$ and η_i=3.

Growth rate (ITG+TEM) H1-NF



The growth rate γ (= γ/ω_{*e}) and the eigen frequency Ω (=ω/ω_{*e}) of the most unstable modes versus b on the magnetic surface $s = 0.7$, $\theta_\kappa = 0$, $\tau = 1$, $\varepsilon_n = 0.1$ and η_i=3.

Growth rate (ITG+TEM)
W7-X



Summary

- The VMEC code is used to generate equilibrium data for the nested magnetic surfaces.
- Mapper code is used to transform data into Boozer coordinates.
- Equilibrium quantities are calculated on magnetic surfaces.
- Ballooning formalisim & WKB type BC are used.
- Shooting technique is applied to find the eigen values.
- i-delta, ITG, TEM models are sucessfully implemented.
- The eigenfunctions are localized within helical wells of B field.
- In W7-X growthrate is smaller than H1-NF.

Few Ref of Published Work

- **Geometrical effects on drift wave instability in stellarator plasmas.**

  Plasma Physics and Controlled Fusion, **46**, 193(2004)

  (**M. H. Nasim**, T. Rafiq, and M. Persson)

- **Dissipative trapped electron modes in a Heliac**

  Physica Scripta, **72**, 409-418(2005)

  (**M. H. Nasim** and M. Persson)

- **Ion temperature gradient driven modes for tokamaks and stellarators**

  30th European Physical Society Conference on"Controlled Fusion and Plasma

  Physics", St. Petersburg, July 7-11,2003, Vol. **27A**, Paper No. 3.9

  Editted by, R. Koch, S. Lebedev

  http://epsppd.epfl.ch/StPetersburg/start.html

  (T. Rafiq, **M. H. Nasim** and M. Persson)

Few Ref of Published work

- **Disstipative trapped electron modes in stellarator plasmas**

  "14th International Stellarator Workshop" in Greifswald, Germany, September22-26, 2003.

  http://www.ipp.mpg.de/eng/for/veranstaltungen/workshops/stellarator_2003

  (**M. H. Nasim,** T. Rafiq and M. Persson)

- **Computation of equilibrium & drift waves in realistic 3D toroidal geometries**

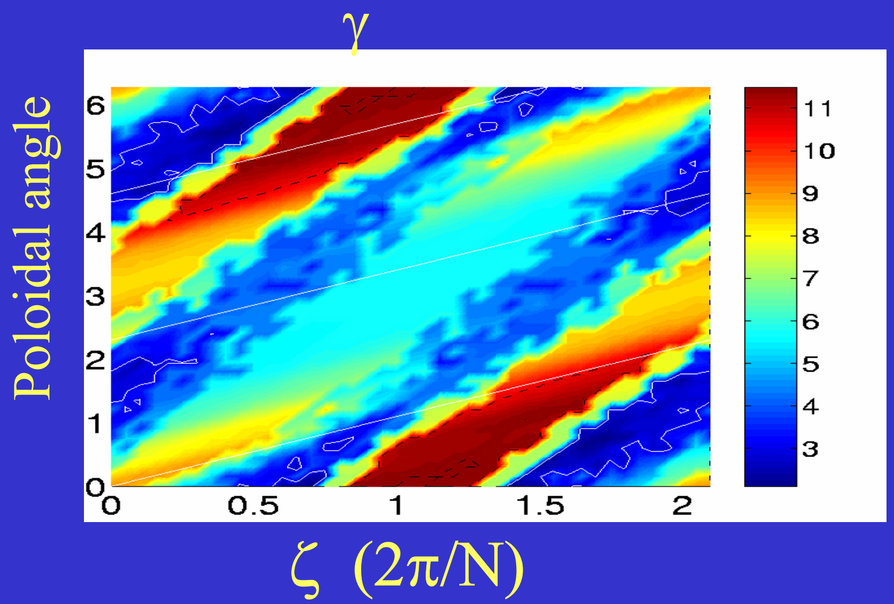
  'IAEA Technical Meeting on Innovative Concepts and Theory of Stellarators", Greifswald, Germany, 29 Sept. to Ist Oct. 2003.

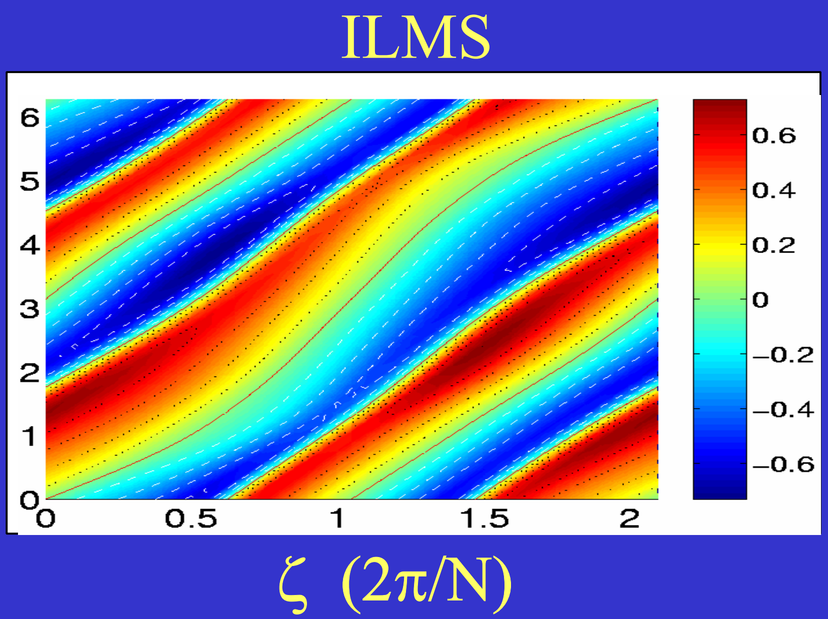
  http://www.ipp.mpg.de/eng/for/bereiche/stellarator/TCM2003/tcm.html

  (**M. H. Nasim,** T. Rafiq and M. Persson)

Thank You

Growth rate in one field period (i-delta) H1-NF





The normalized growth rate γ=($R\gamma/C_s$)the most unstable modes on the magnetic surface s =3/4, θ_κ=0,δ=0.001, ε_n=0.1

Boundary conditions

At large $\zeta = \zeta_{max}$

$$\Phi = \Phi_o exp\left(\varphi(\zeta, \Omega)\right) \qquad (c1)$$

For outgoing wave φ satisfies :

$$Im\left(\varphi'(\pm\zeta_{max}, \Omega)\right) \lessgtr 0, \quad Re\left(\varphi(\pm\zeta_{max}, \Omega)\right) \approx 0, \qquad (c2)$$

For standing wave φ satisfies :

$$Im\left(\varphi(\pm\zeta_{max}, \Omega)\right) \approx 0, \quad Re\left(\varphi(\pm\zeta_{max}, \Omega)\right) \lessgtr 0, \qquad (c3)$$

The eigenvalue equation:

$$\frac{d^2\Phi}{d\zeta^2} + U(\zeta, \Omega)\Phi = 0$$

Using the above Eq.(c1)

$$(\varphi')^2 + \varphi'' + U = 0. \qquad (c4)$$

continue

Boundary conditions

The solutions of this nonlinear equation up to the first order in smallness parameter φ''/φ' are given by the equations

$$\varphi' = \pm i\sqrt{U} - \frac{1}{4}(\ln(U))'$$

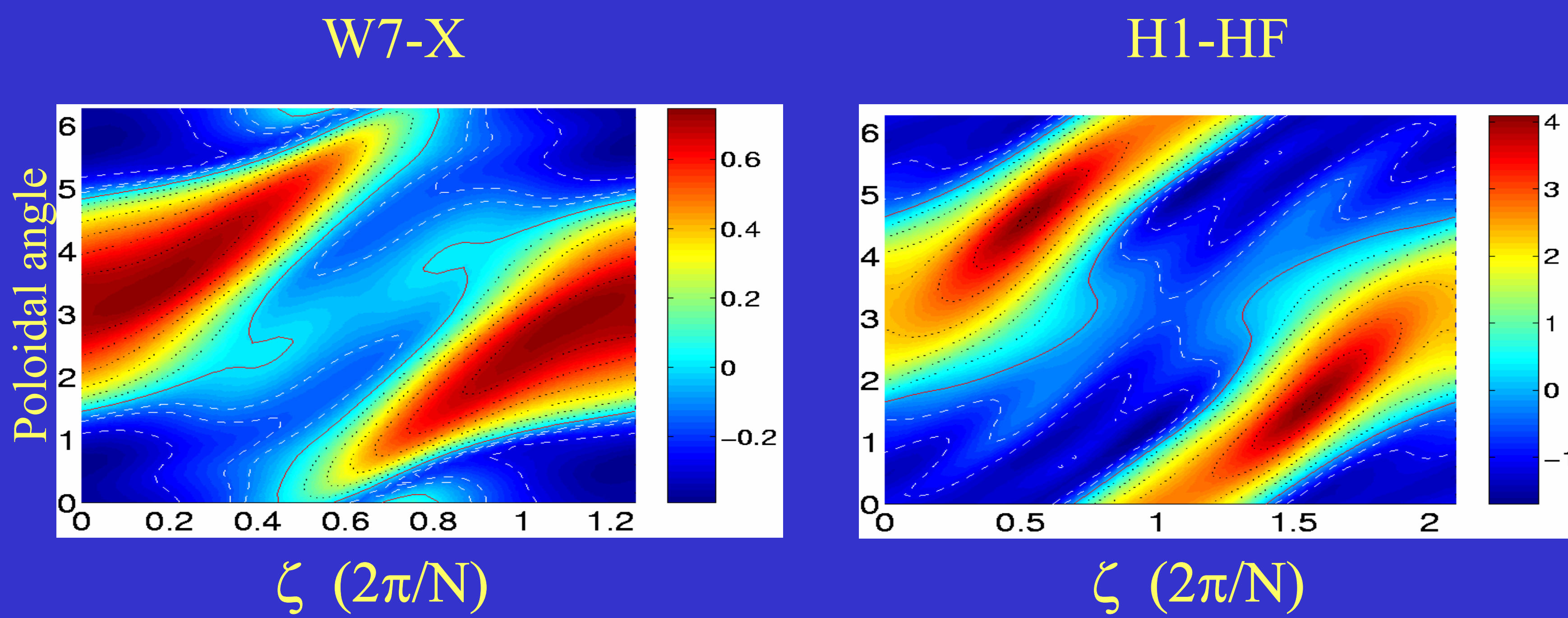
Substituting this in Eq.(c1), one obtains

$$\Phi' = \left[\pm i\sqrt{U} - \frac{1}{4}(\ln(U))'\right]\Phi$$

These are the standard WKB boundary conditions in differential form.
The solution fulfilling condition (c2,c3) is given by

$$\left.\begin{aligned}
\Phi' &= \left[-i|Re\left(\sqrt{U}\right)| + \text{sign}\left(Re\{\sqrt{U}\}\right) Im\{\sqrt{U}\} - \tfrac{1}{4}(\ln(U))'\right]\Phi \ \text{ at } \zeta = +\zeta_{max} \\
&= \left[i|Re\{\sqrt{U}\}| - \text{sign}\left(Re\{\sqrt{U}\}\right) Im\{\sqrt{U}\} - \tfrac{1}{4}(\ln(U))'\right]\Phi \ \text{ at } \zeta = -\zeta_{max} \\
&\text{when } Re\{\sqrt{U}\}(\pm\zeta_{max}) \approx 0, \\
\Phi' &= \left[-|Im\{\sqrt{F}\}| + i\,\text{sign}\left(Im\{\sqrt{U}\}\right) Re\{\sqrt{U}\} - \tfrac{1}{4}(\ln(U))'\right]\Phi \ \text{ at } \zeta = +\zeta_{max} \\
&= \left[|Im\{\sqrt{U}\}| - i\,\text{sign}\left(Im\{\sqrt{U}\}\right) Re\{\sqrt{U}\} - \tfrac{1}{4}(\ln(U))'\right]\Phi \ \text{ at } \zeta = -\zeta_{max}
\end{aligned}\right\}$$

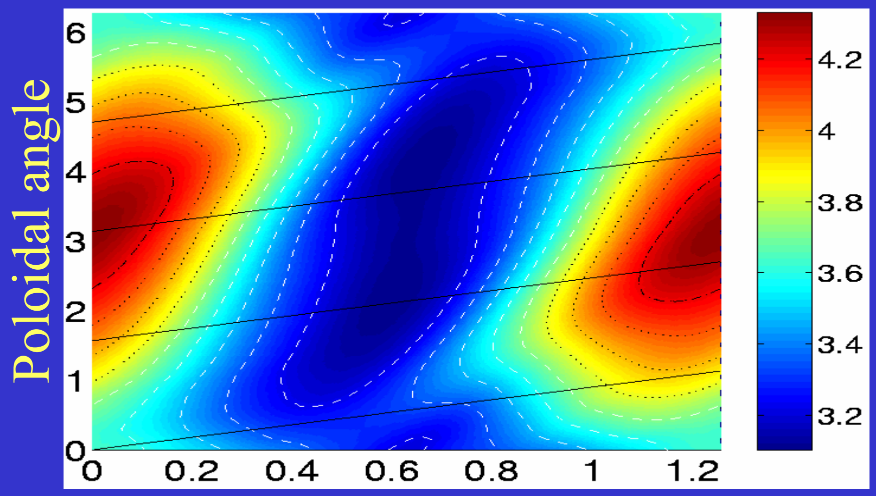
Normal curvature in one field period



Solid line represents the null value

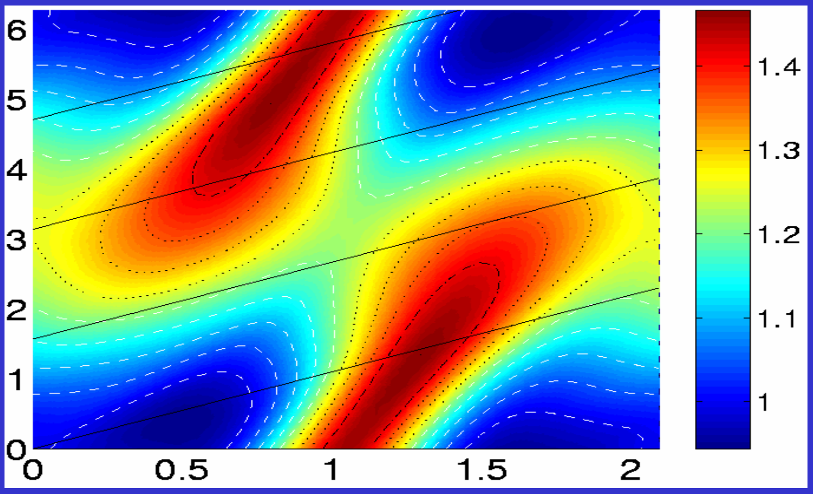
mod B in realistic 3D toroidal geometries

W7-X



ζ $(2\pi/N)$

H1-HF

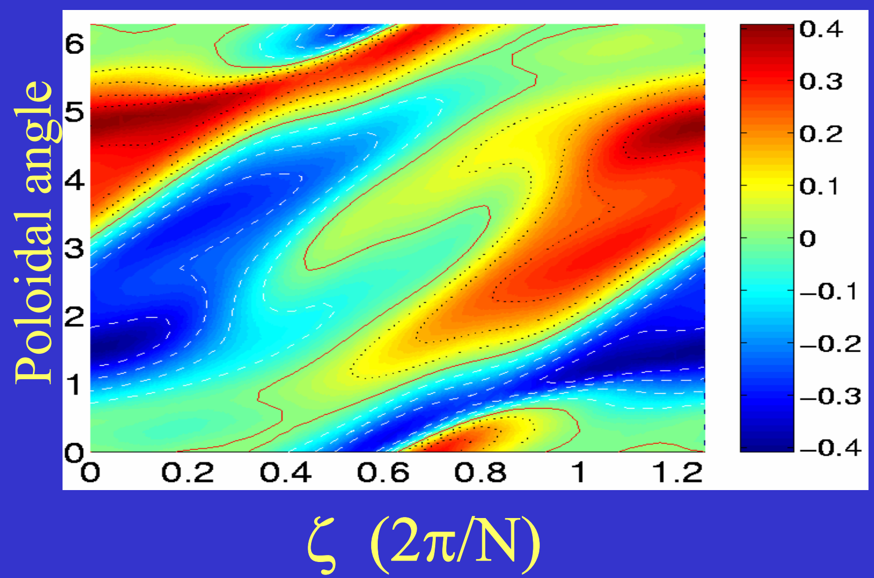


ζ $(2\pi/N)$

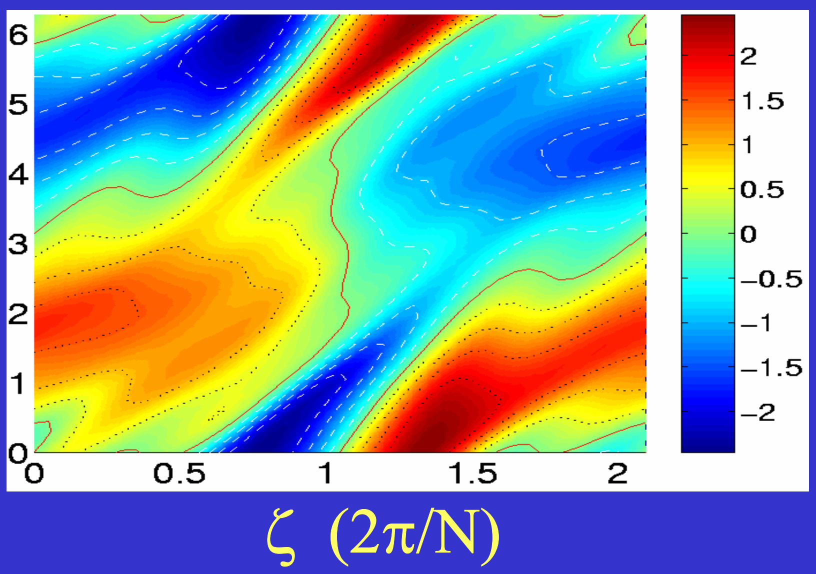
Striaght line represents the field line label

Geodesic curvature in one field period

W7-X

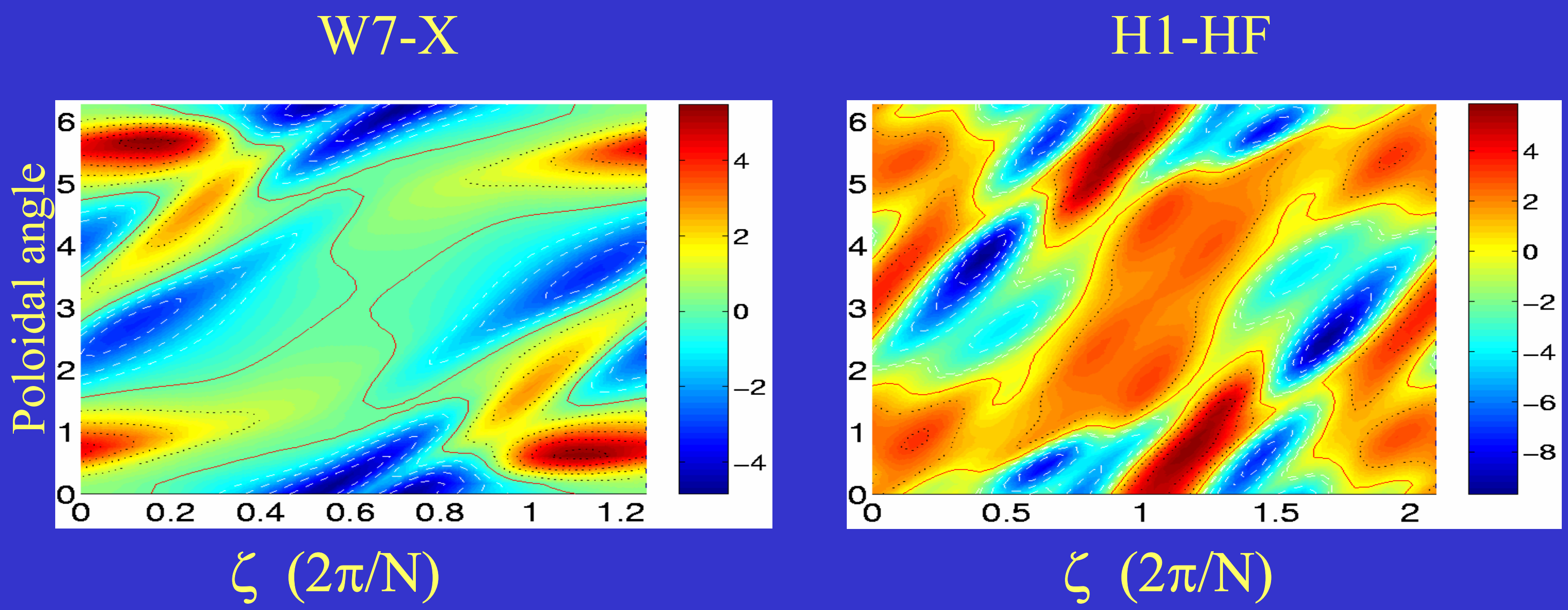


H1-HF



Solid line represents the null value

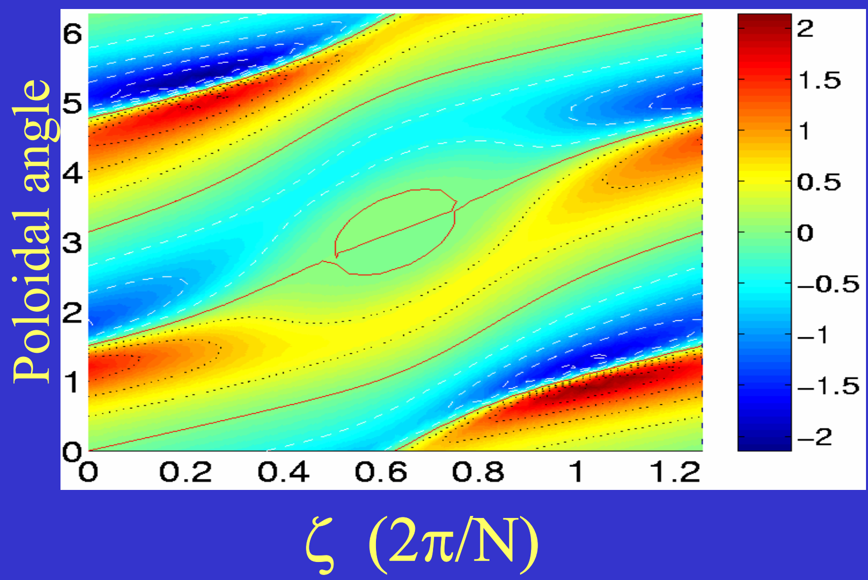
Local Magnetic Shear in one field period



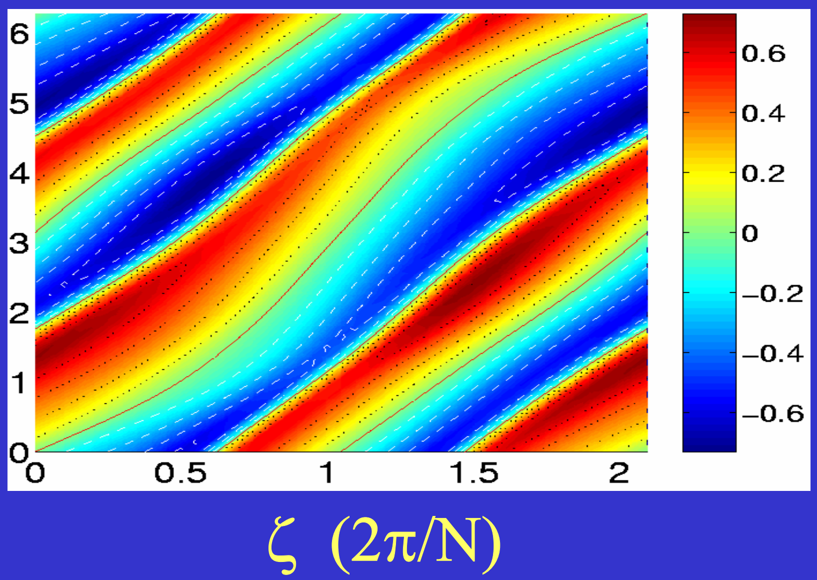
Solid line represents the null value

Integrated Local Magnetic Shear in one field period

W7-X

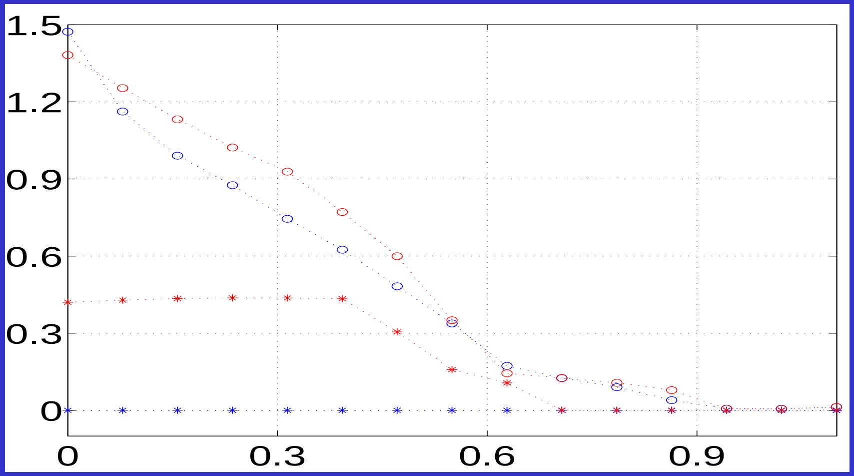


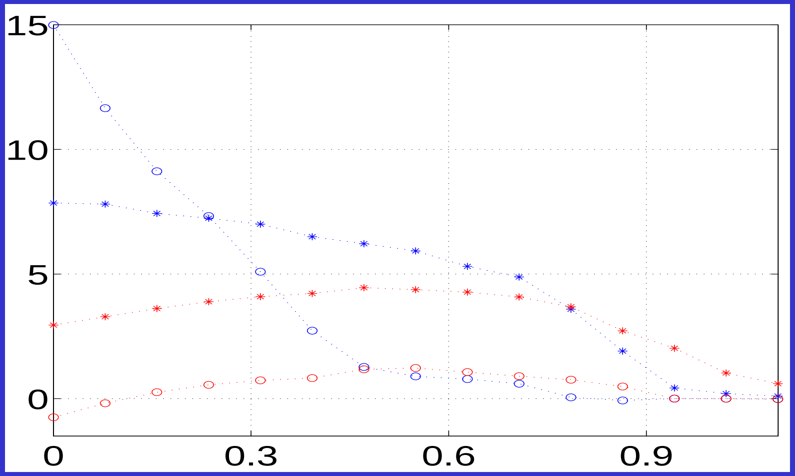
H1-HF



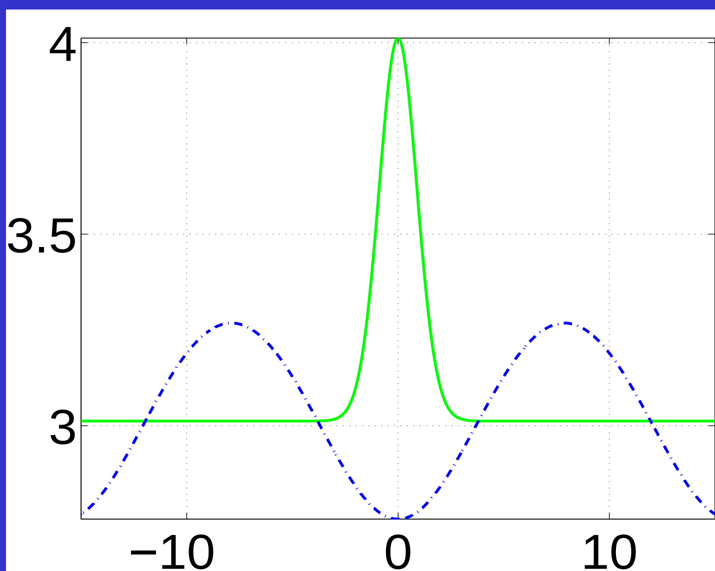
Solid line represents the null value

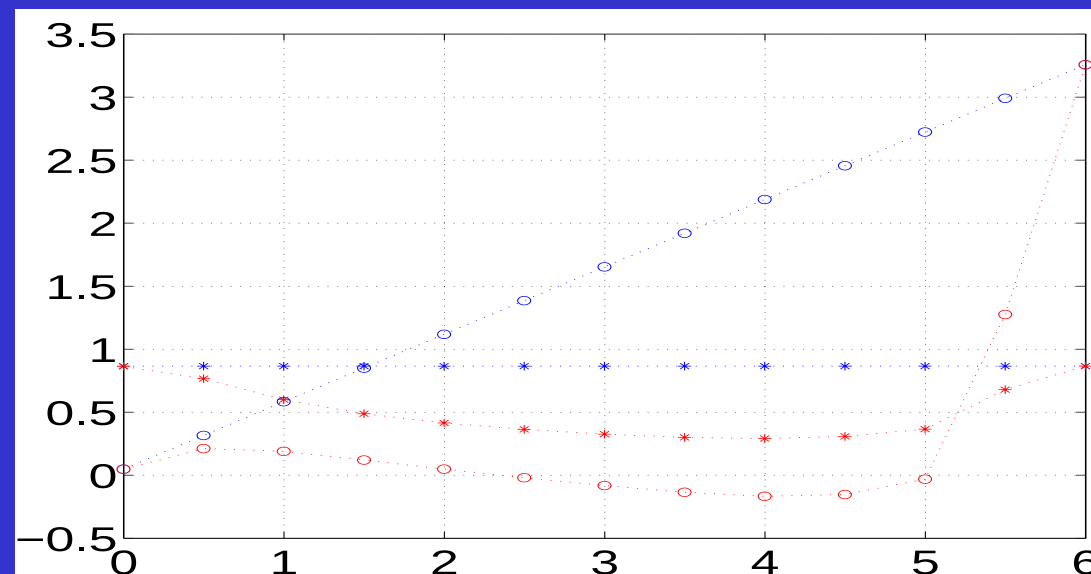
Effect of $\square_k$





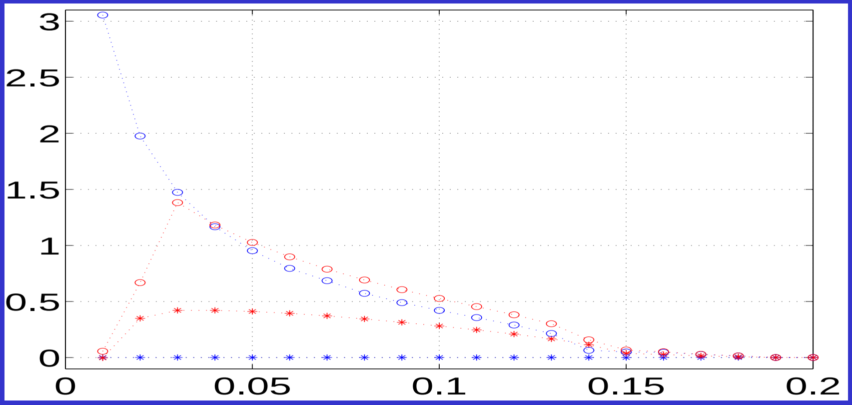
Effect of ♒$_e$

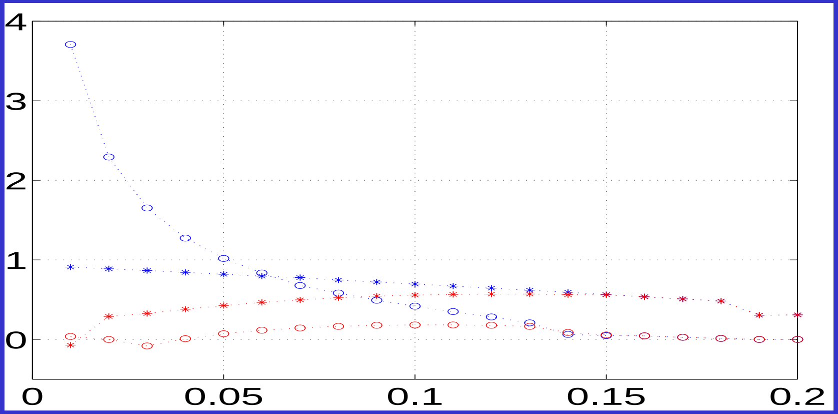




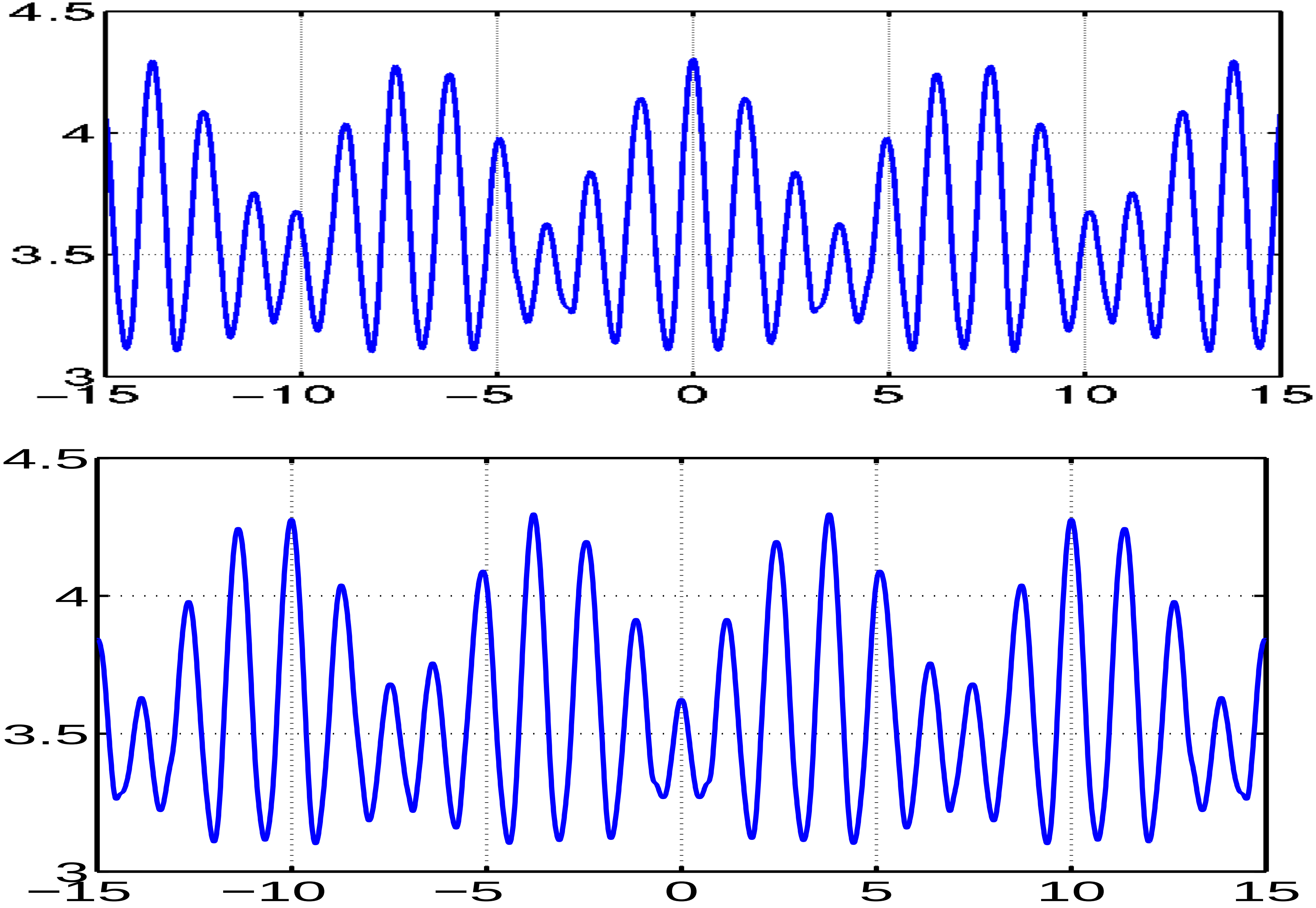
Growth rate as a function of ♒

Effect of Ω_n





Magnetic Field along Field-Line in W7-X



Magnetic Field along Field-Line in H1-NF