QUATERNIONIC AND OCTONIONIC STRUCTURES OF EXCEPTIONAL LIE ALGEBRAS

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Outline

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Division Algebras with Cayley-Dickson Procedure

Let p, q, r, s denote the elements of division algebra other than the octonions.

Then the pairs (p, q) and (r, s) with the multiplication rule

$$(p, q)(r, s) = (pr - s\bar{q}, rq + \bar{p}s)$$

constitute the elements of a division algebra in higher dimension.

The Hurwitz's theorem

There are only four division algebras

- real numbers

$$a, b, c, d \in R$$

- complex numbers

$$\alpha = (a, b) = a + ib\,,\ \beta = (c, d) = c + id\,,\ i^2 = -1$$

$$\alpha, \beta \in C$$

- quaternions

$$q = (\alpha, \beta) = \alpha + \beta j\,, r = (\gamma, \delta) = \gamma + \delta j\,,$$

$$ij = k, i^2 = j^2 = k^2 = ijk = -1$$

$$q, r \in H$$

The Hurwitz's theorem

- octonions

Let $e_1 = i, e_2 = j, e_3 = k$

$$p = (q, r) = q + e_7 r = \sum_{i=0}^{7} p_i e_i$$

$$e_i e_j = -\delta_{ij} + \phi_{ijk} e_k, \qquad i, j, k = 1, 2, \ldots, 7$$

$$\phi_{123} = \phi_{246} = \phi_{435} = \phi_{367} = \phi_{651} = \phi_{572} = \phi_{714} = 1$$

Quaternionic Root Systems and F_4

$SU(2)$: $\pm 1, 0$; $\pm\frac{1}{2}$

$SO(4)$: $(\pm 1, 0) = \pm 1$; $(0, \pm 1) = \pm e_1$

$SO(5)$: $(\pm\frac{1}{2}, \pm\frac{1}{2}) = \frac{1}{2}(\pm 1 \pm e_1)$, $\{\pm 1, \pm e_1\}$

Roots of $SO(8)$:

$SO(8):(SO(5), SO(5)) \Rightarrow$

$T = \{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\}$: **Binary tedrahedral group**

Quaternionic Root Systems and F_4

Define $T = V_0 \oplus V_+ \oplus V_-$: 24-cell $\{3,4,3\}$

$V_0 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$: quaternion group

V_0 : hyperoctahedron

$V_+ = \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$: even #(+) sign, $\quad V_- = \overline{V}_+$,

$V_+ \oplus V_-$: hypercube

Quaternionic Root Systems and F_4

Weights of 8-dimensional representations of $SO(8)$:

$$V_1 = \{\frac{1}{2}(\pm 1 \pm e_1), \frac{1}{2}(\pm e_2 \pm e_3)\}$$

$$V_2 = \{\frac{1}{2}(\pm 1 \pm e_2), \frac{1}{2}(\pm e_3 \pm e_1)\}$$

$$V_3 = \{\frac{1}{2}(\pm 1 \pm e_3), \frac{1}{2}(\pm e_1 \pm e_2)\}$$

$T' = \sqrt{2}(V_1 \oplus V_2 \oplus V_3)$: Dual of T

$O = T \oplus T'$: Binary octahedral group

Roots of $F_4 = T \oplus T' / \sqrt{2}$

The group $Aut(F_4)$

Define the transformations:

$[p,q]:\quad r \to r' = prq;\quad p,q$ unit quaternions $p\bar{p} = q\bar{q} = 1$

$[p,q]^*:\quad r \to r' = p\bar{r}q$

$[p,q] \in SO(4)$, $[p,q]^* \in O(4)/SO(4)$

The Weyl group $W(F_4)$ of order 1152 :

$[T,T] \oplus [T',T'] \oplus [T,T]^* \oplus [T',T']^*$

The group $Aut(F_4)$

Graph automorphism of F_4 can be obtained by the generator γ:

$$1 \leftrightarrow \frac{1}{\sqrt{2}}(1+e_1) \quad , \; e_1 \leftrightarrow \frac{1}{\sqrt{2}}(1-e_1)$$

$$e_2 \leftrightarrow \frac{1}{\sqrt{2}}(e_2-e_3), \quad e_3 \leftrightarrow \frac{1}{\sqrt{2}}(e_2+e_3),$$

$$V_0 \leftrightarrow V_1$$

$Aut(F_4) \approx W(F_4) : \gamma$, (: semi-direct product)

$Aut(F_4) = \{[O,O] \oplus ([O,O]^{*}\}$ of order 2304

The group $Aut(F_4)$

$$W(SO(9)) = \{[V_0,V_0] \oplus [V_+,V_+] \oplus [V_-,V_-] \oplus [V_0,V_0]^* \oplus [V_+,V_+]^* \oplus [V_-,V_-]^* \oplus$$

$$[V_1,V_1] \oplus [V_2,V_3] \oplus [V_3,V_2] \oplus [V_1,V_1]^* \oplus [V_2,V_3]^* \oplus [V_3,V_2]^* \}$$

$$W(SO(8)) = \{[V_0,V_0] \oplus [V_+,V_+] \oplus [V_-,V_-]) \oplus [V_0,V_0]^* \oplus [V_+,V_+]^* \oplus [V_-,V_-]^* \}$$

Embedding $W(SO(9))$ in $Aut(F_4)$

- $W(SO(9))$ can be embedded in $W(F_4)$ triply symmetric way by permuting e_1, e_2, e_3 by cyclic permutation.

- $W(SO(9))$ is the automorphism group of the hyperoctahedron in 4-dimensions of order 384.

- $W(SO(9))$ can be embedded in $Aut(F_4)$ six different ways by a cyclic symmetry C_6 represented by coset element $[V_+, V_1]^6 = [V_0, V_0]$. In each, one of the hyperoctahedra $V_i, i = 0,+,-,1,2,3$ is left invariant.

- The $Aut(F_4)$ is the largest crystallographic group in 4-dimensions and has many interesting non-parabolic subgroups.

- $O / V_0 \approx S_3$

Embedding $W(SO(9))$ in $Aut(F_4)$

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

Table1. Multiplication table of the binary octahedral group

Magic Square

	A_2	C_3	F_4
A_2	$A_2 \oplus A_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

Table 2. The Magic Square

Construction of E_8 root system with quaternions

Define the pairing $\{F_4, F_4\}$:

$$\{T,0\} = T, \quad \{0,T\} = \sigma T \quad ; \qquad \sigma = \frac{1-\sqrt{5}}{2}, \ \tau = \frac{1+\sqrt{5}}{2}$$

$$\{V_1, V_2\} = V_1 + \sigma V_2$$

$$\{V_2, V_3\} = V_2 + \sigma V_3$$

$$\{V_3, V_1\} = V_3 + \sigma V_1$$

Construction of E_8 root system with quaternions

The above quaternions represent roots of E_8 provided the scalar product

$$(p,q)_Q = \frac{1}{2}(p\bar{q} + q\bar{p}) = a + \sigma b \quad \text{replaced by } (p,q)_E = a$$

$$\phi(E_8) = I + \sigma I$$

I : Binary icosahedral group representing the root system of H_4.

$$W(H_4) = \{[I,I] \oplus [I,I]^*\}$$

$W(H_4) \subset W(E_8)$, $\quad |W(H_4)| = 14400$

$W(H_3) \subset W(D_6)$, $\quad |W(H_3)| = 120$

$W(H_2) \subset W(A_5)$, $\quad |W(H_2)| = 10$

Construction of E_8 root system with quaternions

Coxeter Element of $W(H_4) \subset W(E_8)$

$$\left|\lambda I - M\right| = \lambda^8 + \lambda^7 - \lambda^5 - \lambda^4 - \lambda^3 + \lambda^2 + 1 = p(\lambda)q(\lambda) = 0$$

$$p(\lambda) = \lambda^4 + \tau\lambda^3 + \tau\lambda^2 + \tau\lambda + 1 = 0 \rightarrow \mathbf{exp}(im\frac{2\pi}{30})$$

$m = 7,13,17,23$

$$p(\lambda) = \lambda^4 + \sigma\lambda^3 + \sigma\lambda^2 + \sigma\lambda + 1 = 0 \rightarrow \mathbf{exp}(im\frac{2\pi}{30})$$

$m = 1,11,19,29$ (Coxeter exponents of $W(H_4)$)

Maximal Subgroups of $W(H_4)$:

1. $Aut(A_2 \oplus A_2)$ of order 144
2. $Aut(H_2 \oplus H_2)$ of order 400
3. $W(H_3 \oplus A_1)$ of order 240
4. $Aut(A_4)$ of order 240
5. $W(SO(8)):C_3$ of order 576

$W(H_3)$ is the symmetry of the molecule C_{60} and the quasicrystals possessing five-fold and three-fold symmetries.

More on this when we discuss polyhedra obtained from 120-cell and 600-cell invariant under $W(H_4)$.

Octonionic Root System of E_8

Cayley-Dickson procedure with (F_4, F_4) Magic Square $\Rightarrow$

$(T,0) = T$, $(0,T) = e_7 T$

$(V_1, V_1) = V_1 + e_7 V_1$, $(V_2, V_3) = V_2 + e_7 V_3$, $(V_3, V_2) = V_3 + e_7 V_2$

240=24+24+64+64+64

240 octonionic roots of E_8 are closed under multiplication

Imaginary elements of these sets constitute the 126 roots of E_7

Automorphism group of 126 roots of E_7 is the Chevalley group $G_2(2)$ of order 12096

Octonionic Root System of E_8

Any octonion $s = H + e_7 H'$ has a symmetry which preserves the quaternionic decomposition $s \to t = pH\bar{p} + e_7 pH'q$ where $[p,q]$ An element of $SO(4) \subset G_2$.

The finite subgroup of $SO(4)$ leaving E_7 root system is a group of order 192(17):
$[T,V_0] \oplus [T',V_1] = [V_0,V_0] \oplus [V_+,V_0] \oplus [V_-,V_0] \oplus [V_1,V_1] \oplus [V_2,V_1] \oplus [V_3,V_1]$

One can prove that such a structure can be obtained 63 different ways so that the order of the automorphism group is 192x63=12096.

Maximal subgroups of $G_2(2)$ of orders 432,336 and 192(14) are the automorphism groups of the root systems of the maximal subalgebras $E_6, SU(8)$ and $SO(12) \oplus SU(2)$ respectively.

The Group $Z_2^{\,3}.PSL_2(7)$ of order 1344

Automorphism group of the set quaternions

$$A=\{\pm e_1,\pm e_2,\pm e_3,\pm e_4,\pm e_5,\pm e_6,\pm e_7\}=\mathbf{Im}\,V_0+e_7V_0$$

Remember $W(SO(9))\subset O(4)$ leaves one of the set V_i invariant. The subgroup of $W(SO(9))/Z_2$ in $SO(4)$ of order 192 leaving the structure $\mathbf{Im}\,V_0+e_7V_0$ invariant:

$$W(SO(9))/Z_2=\{[V_0,V_0]\oplus[V_+,V_-]\oplus[V_+,V_-]\oplus[V_1,V_1]\oplus[V_2,V_2]\oplus[V_3,V_3]\}$$

The set $\mathbf{Im}\,V_0=\{\pm e_1,\pm e_2,\pm e_3\}$ can be chosen 7 different ways from the set A. Therefore the group leaving the octonions A is of order 7x192=1344.

It is the non-split extension of the elementary abelian group $Z_2^{\,3}$ by the Klein's group $PSL_2(7)$ of order 168.

Maximal Subgroups of $Z_2^3.PSL_2(7)$

- The group 192(13); Rep. 7=3+4
- The group 192(14); Rep. 7=1+6
- The group $Z_2^3 : Z_3 : Z_7$ of order 168
  Rep. 7=7

Quaternions and Polyhedra

The unit quaternions $q\bar{q} = 1$ define the unit sphere S^3.

When cut by the hyperplane $q_0 = const$ one obtains the sphere S^2 described by $q_1^2 + q_2^2 + q_3^2 = 1 - q_0^2$ preserved by $O(3) \approx Z_2 \times SO(3)$.

The group elements : $[p, \pm\bar{p}],\ p \in S^3$

The polytope in 4-dimension $\{3,4,3\} = T$, Dual of T is $T' = \{3,4,3\}$.

The polytope 600-cell is described by the quaternions of binary icosahedral group $I = \{3,3,5\}$ with 120 vertices.

Quaternions and Polyhedra

The conjugacy classes of $O = T + T'$ and the I describe the ployhedra in 3- dimension.

- For O the group $[p, \pm\bar{p}]$ is the octahedral group with inversion, $p \in O$.
- For I the group $[p, \pm\bar{p}]$ is the icosahedral group with inversion isomorphic to the Coxeter group H_3.
- The dual of $\{3,3,5\}$ is a polytope called 120-cell with 600 vertices described by unit quaternions. The orbits of H_3 in 120-cell define the polyhedra in 3-dimensions.

Quaternions and Polyhedra

I. Orbits of octahedral group in $O = T + T'$

$q_0 = \pm\frac{1}{2}$, the vertices $\frac{1}{2}(\pm 1 \pm 1 \pm 1)$ form a cube

$q_0 = \pm\frac{1}{\sqrt{2}}$, the vertices $\{\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\}$ form a octahedron

$q_0 = 0$, the vertices $\{\pm 1, \pm 1, \pm 1\}$ form a octahedron

$q_0 = 0$, the vertices $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0), \frac{1}{\sqrt{2}}(\pm 1, 0, \pm 1), \frac{1}{\sqrt{2}}(0, \pm 1, \pm 1)$

form a truncated octahedron.

Quaternions and Polyhedra

II. Orbits of H_3 in the polytope $I = \{3,3,5\}$

$q_0 = \pm\frac{\tau}{2}$, the vertices $\frac{1}{2}(\pm 1,0,\pm\sigma), \frac{1}{2}(0,\pm\sigma,\pm 1), \frac{1}{2}(\pm\sigma,\pm 1,0)$ form a icosahedron.

$q_0 = \pm\frac{\sigma}{2}$, the vertices $\frac{1}{2}(\pm 1,\pm\tau,0), \frac{1}{2}(0,\pm 1,\pm\tau), \frac{1}{2}(\pm\tau,0,\pm 1)$ form a icosahedron.

$q_0 = \pm\frac{1}{2}$, the vertices $\frac{1}{2}(\pm 1,\pm 1,\pm 1)$,

$\frac{1}{2}(\pm\tau,\pm\sigma,0)$, $\frac{1}{2}(0,\pm\tau,\pm\sigma), \frac{1}{2}(\pm\sigma,0,\pm\tau)$ form a dodecahedron.

$q_0 = 0$, the vertices $\{\pm 1,\pm,\pm, \frac{1}{2}(\pm\sigma,\pm\tau,\pm),$

$\frac{1}{2}(\pm 1,\pm\sigma,\pm\tau), \frac{1}{2}(\pm\tau,\pm 1,\pm\sigma)\}$ form a icosidodecahedron with 30 vertices.

Quaternions and Polyhedra

III.120-cell {5,3,3} with 600 vertices

The vertices of {5,3,3} can be constructed from the 24 vertices of T' by

$$\{5,3,3\} = \sum_{i=0}^{4}\sum_{j=0}^{4} \oplus\, p^i T' \bar{p}^j \,,\ p \subset I \,,\ p^5 = \pm 1,$$

24x25=600

H_3 has 15 orbits in {5,3,3}

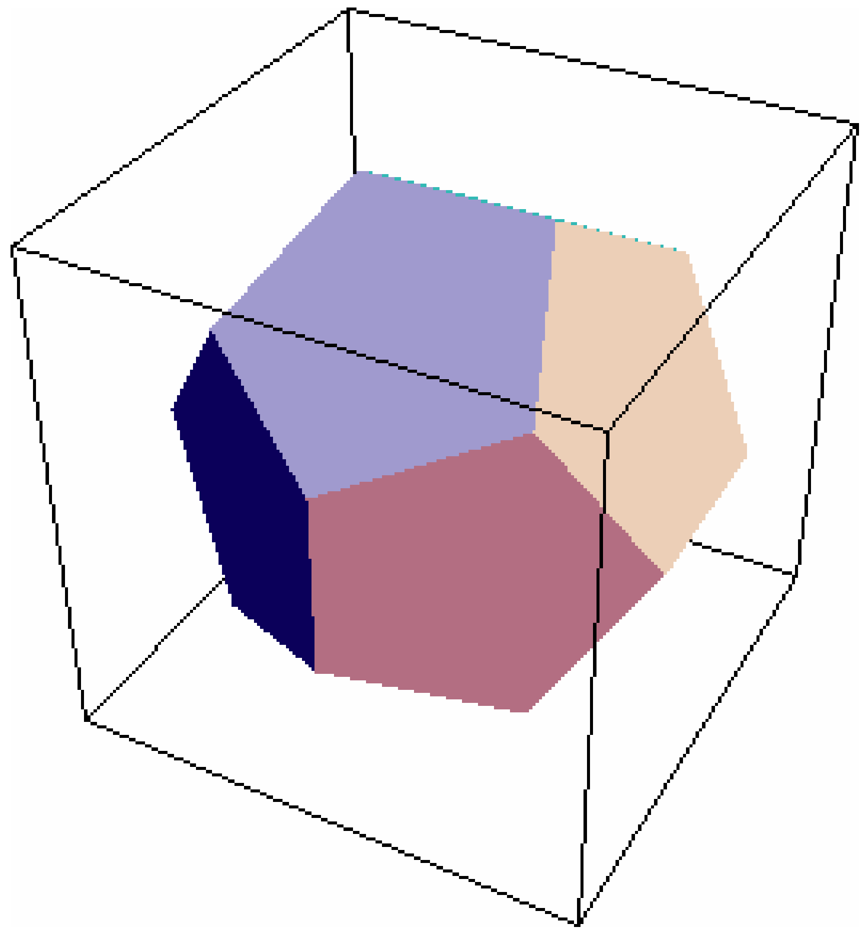
Quaternions and Polyhedra

$$1.\ q_0 = \pm\frac{\tau^2}{2\sqrt{2}},$$

$$\frac{1}{2\sqrt{2}}(0,\pm 1,\pm\sigma^2) + cyclic\,perm.,$$

$$\frac{\sigma}{2\sqrt{2}}(\pm 1,\pm 1,\pm 1)$$

Dodecahedron with 20 vertices:



Quaternions and Polyhedra

2. $q_0 = \pm \frac{\sigma^2}{2\sqrt{2}}$,

$$\frac{1}{2\sqrt{2}}(0, \pm 1, \pm \tau^2) + cyclicperm., \frac{\tau}{2\sqrt{2}}(\pm 1, \pm 1, \pm 1)$$

Another dodecahedron

3. $q_0 = \pm \frac{\tau - \sigma}{2\sqrt{2}}$,

$$\frac{1}{2\sqrt{2}}(0, \pm \tau, \pm \sigma^2) + \text{cyclic perm.}, \ \frac{1}{2\sqrt{2}}(\pm 1, \pm 1, \pm 1)$$

Another dodecahedron

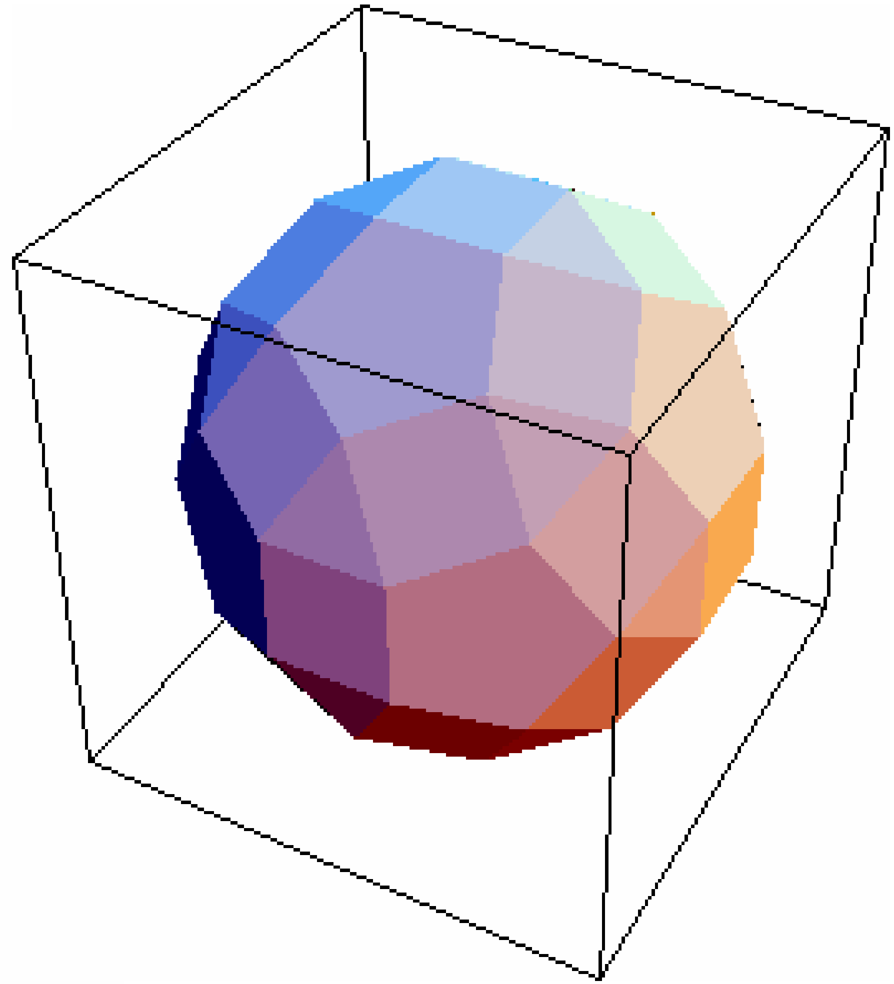
Quaternions and Polyhedra

4. $q_0 = \pm\frac{\sigma}{2\sqrt{2}}$; Small Rhombicosidodecahedron

$\frac{1}{2\sqrt{2}}(0, \pm(\tau-\sigma), \pm\tau)$ + cyclic perm. ; 12 vertices

$\frac{1}{2\sqrt{2}}(\pm 2, \pm 1, \pm\tau)$ + cyclic perm. ; 24 vertices

$\frac{1}{2\sqrt{2}}(\pm\sigma, \pm\sigma, \pm\tau^2)$ + cyclic perm. ; 24 vertices



Quaternions and Polyhedra

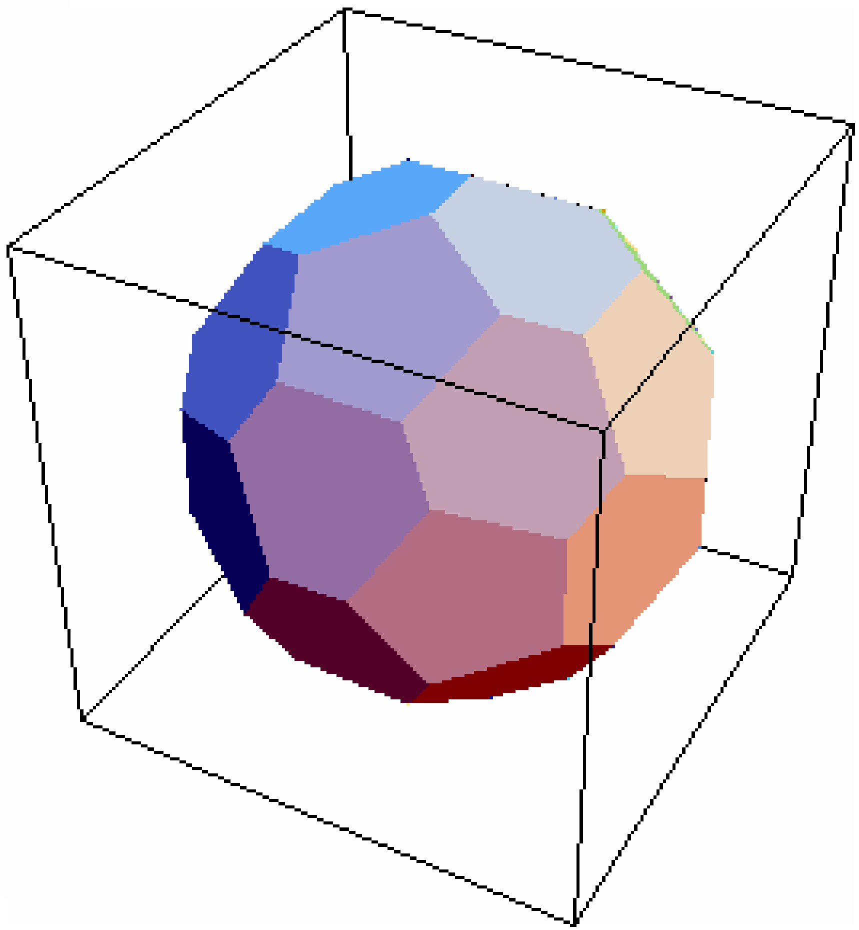
5. $q_0 = \pm\frac{1}{2\sqrt{2}},$

$\frac{1}{2\sqrt{2}}(0,\pm\sigma^2,\pm\tau^2)+$ cyclic perm. ; 12 vertices

$\frac{1}{2\sqrt{2}}(\pm1,\pm1,\pm(\tau-\sigma))+$ cyclic perm. ; 24 vertices

$\frac{1}{2\sqrt{2}}(\pm\sigma,\pm2,\pm\tau)+$cyclic perm. ; 24 vertices

C_{60}??

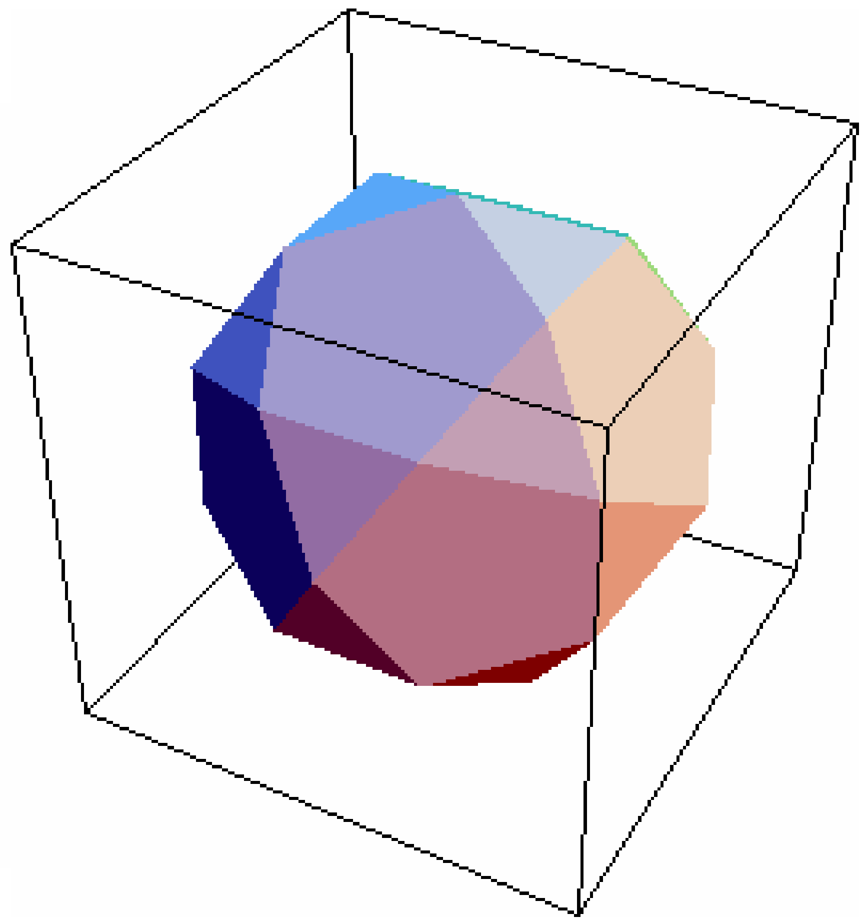


Quaternions and Polyhedra

$6.\ q_0 = \pm\frac{\tau}{2\sqrt{2}},$ Icosidodecahedron

$\frac{1}{\sqrt{2}}(\pm1,0,0),\ \frac{1}{\sqrt{2}}(0,\pm1,0),\frac{1}{\sqrt{2}}(0,0,\pm1)\ :6$

$\frac{1}{2\sqrt{2}}(\pm1,\pm\sigma,\pm\tau)+$ cyclic perm. $:24$



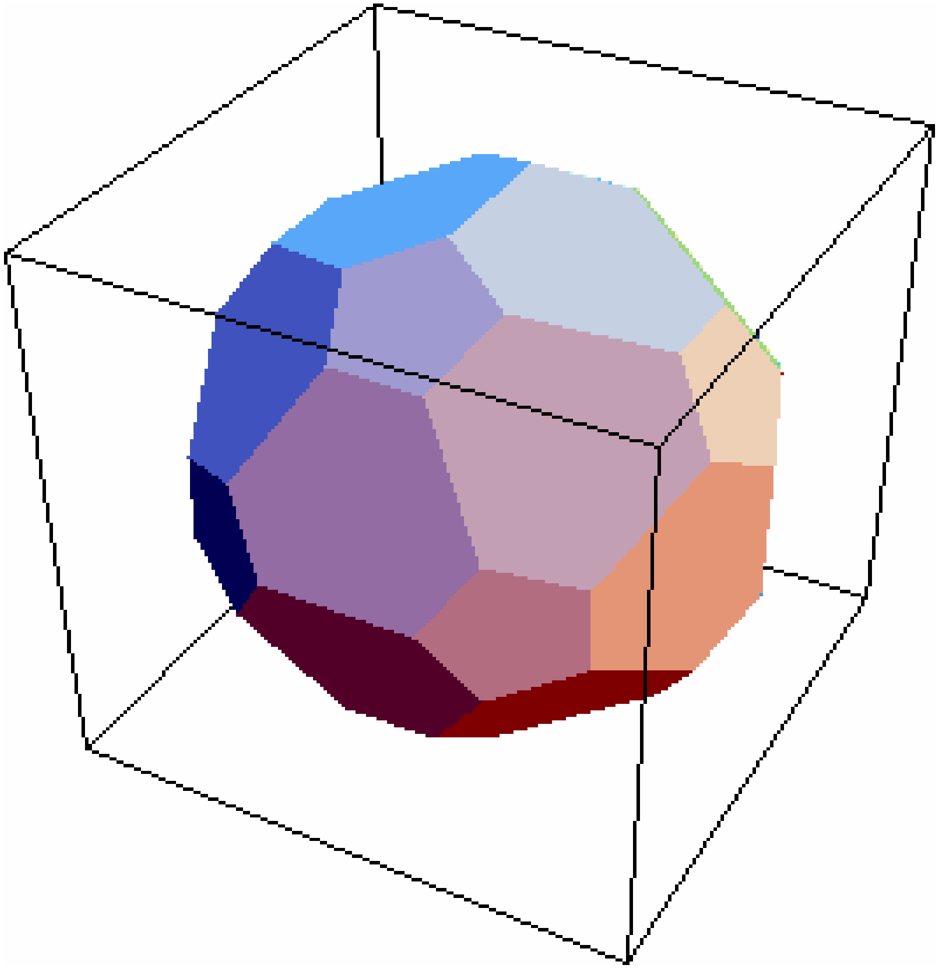
Quaternions and Polyhedra

7. $q_0 = \pm\frac{\tau}{2\sqrt{2}}$,

$\frac{1}{2\sqrt{2}}(0,\pm\sigma,\pm(\tau-\sigma))$+cyclic perm.: 12

$\frac{1}{2\sqrt{2}}(\pm1,\pm2,\pm\sigma)$ + cyclic perm. : 24

$\frac{1}{2\sqrt{2}}(\pm\tau,\pm\tau,\pm\sigma^2)$ + cyclic perm.: 24



Quaternions and Polyhedra

8.

$q_0 = 0,$

$\frac{1}{2\sqrt{2}}(0,\pm 1,\pm 1)$ + cyclic perm. : 12

$\frac{1}{2\sqrt{2}}(\pm 1,\pm \tau^2,\pm \sigma^2)$ + cyclic perm. : 24

$\frac{1}{2\sqrt{2}}(\pm(\tau-\sigma),\pm \sigma,\pm \tau)$ + cyclic perm.: 24