Transition amplitudes for time-dependent linear harmonic oscillator with Linear time-dependent terms added to the Hamiltonian.

By

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Abstract

Recently, we obtained the transition amplitudes for a general time-dependent linear harmonic oscillator using standard operator techniques. In this paper, we follow the same techniques, to calculate the transition amplitudes after adding a linear time-dependent term to the Hamiltonian. This allows for odd to even and even to odd transitions which was not allowed when these linear terms were absent.

1. Introduction

Time-dependent harmonic oscillators have been considered by many authors [1 – 4]. From the point of view of a physical application, Parker [5] applied the alpha and beta coefficients of the problem to the cosmological creation of particles in an expanding universe. Earlier, Kanai [2] had considered a simple form of the time-dependent linear oscillator. Through this model was criticized by Brittin [6] and Senitzky [7] for various reasons, Landovitz et al, ignoring the criticism, proceeded to calculate the Green's function [8] for the general form of Kanai's model and used it to calculate the corresponding transition amplitudes [9]. Their calculations are very difficult to comprehend.

Recently we used standard operators to calculate the transition amplitudes for the general time-dependent linear harmonic oscillator in a transparent manner [10]. We anticipate that our approach will be relevant to other physical problems including Senitzky's [7] complex model for the dissipative quantum mechanical oscillator.

In this paper, we have used the same manifest operator techniques to obtain the transition amplitudes after adding linear time-dependent terms to the Hamiltonian which allow for transitions from odd to even and even to odd states. These transitions were not allowed under the even-parity Hamiltonian which did not include the linear terms.

This paper is organised as follows. In section 2, we introduce the modified Hamiltonian and obtain the transformed operator $x_{\pm}(t)$, $p_{\pm}(t)$ in terms of the non-transformed time-independent operators x and p and the coefficients in the

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"transition matrix". In section 3, we calculate the corresponding transformed creation and annihilation operators. In section 4, we derive the recursion relations satisfied by these transition amplitudes. These recursion relations are used in section 5 to calculate a generating function which gives the transition amplitudes in terms of the initial one. This initial one is then evaluated using its relationship with a known identity to complete the calculation.

2. The Transformed Operators $x_\pm(t)$, $p_\pm(t)$

The Hamiltonian for a time-dependent linear harmonic oscillator with linear time-dependent terms added is given by

$$H(t) = f(t)\,\frac{p^2}{2m} + \frac{1}{2}g(t)\,mw^2x^2 + u(t)\,p + v(t)x \quad (1)$$

Note that we have emphasized the t-dependence of the Hamiltonian while we have suppressed the x and p-dependence. This convention will be used in the sequel for all the operators.

In the above, the functions f(t), g(t), u(t) and v(t) are all real continuous functions to make the Hamiltonian hermitian. Also f(t) = g(t) = 1, u(t) = v(t) = 0 gives the usual time-independent linear harmonic oscillator Hamiltonian whereas putting u(t) = v(t) = 0 we arrive at the case previously discussed by us of a time-dependent linear harmonic oscillator without the linear terms.

The wave-functions at an arbitrary time t are related to the ones at time t = 0 through a time-dependent unitary transformation U(t) by

$$\psi(x,t) = U(t)\,\psi(x,0) \qquad (2)$$

where

$$U(t)\,U^+(t) = U^+(t)\,U(t) = I \qquad (3)$$

The Schrodinger equation

$$i\hbar\,\frac{\partial}{\partial t}\psi(x,t) = H(t)\psi(x,t) \qquad (4)$$

and equation (2) give

$$i\hbar\,\frac{\partial}{\partial t}\,U(x,t) = H(t)\;U(x,t) \qquad (5)$$

which for the time-independent Hamiltonian, results in the obviously unitary "formal" solution

$$U(t) = e^{\frac{-i}{\hbar}Ht} \qquad (6)$$

In general, however, equation (5) cannot be solved analytically.

We define the operators $O_{\pm}(t)$ corresponding to any operator O(t) (which may have a manifest time-dependence) by

$$O_{\pm}(t) = U^{+}(t)\, O(t)\, U(t), \quad O_{-}(t) = U(t)\, O(t)\, U^{+}(t) \qquad (7)$$

These operators satisfy the dynamical equations

$$\frac{\partial}{\partial t} O_{+}(t) = \frac{1}{i\hbar}[O_{+}(t), H_{+}(t)] + (\frac{\partial}{\partial t}O)_{-} \qquad (8a)$$

$$\frac{\partial}{\partial t} O_{-}(t) = \frac{1}{i\hbar}[H(t), O_{-}(t)], + (\frac{\partial}{\partial t}O)_{-} \qquad (8b)$$

which have a slight asymmetry. Note that the operator in equation (8b) in the commutator is H(t) and not $H_{-}(t)$ as may be expected if there were symmetry.

The operators $x_{+}(t)$, $p_{+}(t)$ are related to x and p by

$$x_{+}(t) = U^{+}(t)\, x\, U(t) = a(t)\, x + b(t)\, p + y_1(t) \qquad (9a)$$

$$p_{+}(t) = U^{+}(t)\, p\, U(t) = c(t)\, x + d(t)\, p + y_2(t). \qquad (9b)$$

Since

$$[x_{+}(t), p_{+}(t)] = U^{+}(t)\, [x, p]\, U(t) = i\hbar = [x, p], \qquad (10)$$

$$a(t)\, d(t) - b(t)\, c(t) = 1. \qquad (11)$$

Thus the homogeneous part of the transformation in equations (9) is given by a unimodular matrix and in particular, its determinant is t – independent. Note that the presence of the functions $y_1(t)$, $y_2(t)$ in equations (9) expresses the non-homogenous nature of the transformation which is resulting from the linear terms in the Hamiltonian.

To obtain the differential equations system which determines the coefficients a(t), b(t), c(t) d(t), $y_1(t)$, $y_2(t)$ in equations (9) w use

$$\frac{\partial x_+}{\partial t} = \frac{1}{i\hbar}[x_+(t), H_+(t)] = f(t)\frac{p_+(t)}{m} + u(t) \qquad (12)$$

and the expressions for $x_+(t)$, $p_+(t)$ in equation (9) to arrive at

$$\dot{a}(t)\,x + \dot{b}(t)\,p + \dot{y}_1(t) = \frac{f(t)}{m}(c(t)x + d(t)\,p + y_2(t)) + u(t) \qquad (13)$$

which gives

$$\dot{a}(t) = \frac{f(t)}{m}c(t),\ \dot{b}(t) = \frac{f(t)}{m}d(t) \qquad (14)$$

$$\dot{y}_1(t) = \frac{f(t)}{m}y_2(t) + u(t) \qquad (15)$$

Similarly from,

$$\frac{\partial p_-}{\partial t} = \frac{1}{i\hbar}[p_-(t), H_+(t)] = -g(t)mw^2x_+(t) - v(t) \qquad (16)$$

we obtain

$$\dot{c}(t) = -g(t)mw^2a(t), \quad \dot{d}(t) = -g(t)mw^2\,b(t) \qquad (17)$$

$$\dot{y}_2 t) = -g(t)mw^2\ y_1(t) - v(t) \qquad (18)$$

We note that from equations (14, 17)

$$\frac{d}{dt}(a(t)d(t) - b(t)\,c(t)) = 0$$

which shows that a(t) d(t) – c(t) b(t) is independent of time.

Thus

$$a(t)\ d(t) - b(t)\ c(t) = a(0)\ d(0) - b(0)\ c(0) = 1 \qquad (19)$$

Since

$$a(0) = d(0) = 1, \quad b(0) = c(0) = 0 \qquad (20)$$

The result in equation (19) has already been obtained in equation (11) using a different argument.

From equations (14) and (17), we may write second order differential equations for the functions a(t), b(t), c(t), d(t) which are for a(t), b(t)

$$\ddot{Z}(t) - \frac{f'(t)}{f(t)}\dot{Z}(t) + f(t)\,g(t)\,w^2 Z(t) = 0 \qquad (21)$$

and for c(t), d(t)

$$\ddot{Z}(t) - \frac{g'(t)}{g(t)}\dot{Z}(t) + f(t)\,g(t)\,w^2 Z(t) = 0 \qquad (22)$$

Subject to the initial conditions, a(0) = d(0) = 1, b(0) = c(0) = 0, the equations (14, 17) have unique solutions for a(t), b(t), c(t), d(t). For the time-independent case (f(t) = g(t) = 1), these are explicitly given as

$$a(t) = d(t) = \cos wt, \quad b(t) = \frac{1}{mw}\sin wt, \quad c(t) = -\ mw \sin wt \qquad (23)$$

In the absence of the linear terms ie when u(t) = v(t) = 0, the homogeneous system of equations (15) and (19) for $y_1(t)$ $y_2(t)$ have the unique trivial solution

$$y_1(t) = y_2(t) = 0 \qquad (24a)$$

using $y_1(0) = y_2(0) = 0$ (24b)

This is as expected and can be proved easily using standard theorem from theory of system of first order ordinary differential equations.

For completeness, we note that $y_1(t)$, $y_2(t)$ satisfy the non-homogeneous second order differential equations.

$$\ddot{y}_1(t) - \frac{f'(t)}{f(t)}\dot{y}_1(t) + f(t)g(t)\,w^2\, y_1(t) = \dot{u}(t) - (\frac{f'(t)}{f(t)}u(t) + \frac{f(t)}{m}\,v(t)) \qquad -(25a)$$

$$\ddot{y}_2(t) - \frac{g'(t)}{g(t)}\dot{y}_2(t) + f(t)g(t)\,w^2\, y_2(t) = \dot{v}(t) - (\,g(t)mw^2u(t) + \frac{g'(t)}{g(t)}\,v(t)) \qquad -(25b)$$

Finally, we may use similar techniques to obtain the operation $x_-(t)$, $p_-(t)$ in the form

$x_-(t) = d(t)\ x\ -\ b(t)\ p - d(t)\ y_1(t) + b(t)\ y_2(t)$ -(26a)
$p_-(t) = -\ c(t)\ x + a(t)\ p + c(t)\ y_1(t)\ -\ a(t)\ y_2(t)$ (26b)

Note that the homogeneous part in the above equations is given by the matrix $\begin{pmatrix} d(t) & -b(t) \\ -c(t) & a(t) \end{pmatrix}$ which is unimodular and inverse of the matrix $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ in equations (9).

3. <u>Calculation of the transformed creation and annihilation operators</u>

The non – Hermition creation and annihilation operators A^+ and A are related to the Hermition operators x and p through.

$$A^+ = \frac{-i}{\sqrt{2m\hbar w}}(p + imwx) \tag{27a}$$

$$A = \frac{i}{\sqrt{2m\hbar w}}(p - imwx) \tag{27b}$$

The above equations can be inverted to give

$$x = \sqrt{\frac{\hbar}{2mc}}(A + A^+) \tag{28a}$$

$$p = -i\sqrt{\frac{m\hbar w}{2}}\,(A - A^+) \tag{28b}$$

The operators A and A^+ satisfy the commutation relation

$$[A, A^+] = 1. \tag{29}$$

In terms of the energy eigenstates of the Hamiltonian H = H(0) given by

$$H\,|n\rangle = (n + ½)\ \hbar\omega\,|\,n\,\rangle, \tag{30}$$

the operators A and A^+ have the matrix elements

$$< m|\,A|n> = \sqrt{n}\ \delta_{mn-1} \qquad \text{-(31a)}$$

$$< m|\,A^+|n> = \sqrt{n+1}\ \delta_{mn+1} \tag{31b}$$

Next we compute the transformed operators $A_+(t)$ and $A^+_+(t)$ in terms of the function a(t), b(t), c(t), d(t), $y_1(t)$, $y_2(t)$ appearing in the transformed operators

$x_+(t)$ and $p_+(t)$ in equation 9. Indeed using equations (7, 27b, 9, 28) in that order, we have

$$A_+(t) = U^+(t)\; AU(t) = \frac{i}{\sqrt{2m\hbar w}}\, U^+(t)\,(p - imwx)\, U(t)$$

$$= \frac{i}{\sqrt{2m\hbar w}}(p_+(t) - imwx_+(t))$$

$$= \frac{i}{\sqrt{2m\hbar w}}[(c(t) - imwa(t))\, x + (d(t) - imwb(t))p \;+ (y_2(t) - imw\, y_1(t))]$$

$$= \frac{i}{\sqrt{2m\hbar w}}[(c(t) - imwa(t)\sqrt{\frac{\hbar}{2mw}}\;(A + A^+)$$

$$+ \;(d(t) - imw\, b(t))\,(-i\sqrt{\frac{m\hbar w}{2}}(A - A^+))$$

$$+ \;y_2(t) - imw\; y_1(t)$$

$$= \alpha\;(t)\; A + \;\beta\;(t)\; A^+ + \gamma\;\;(t) \qquad (32)$$

where

$$\alpha(t) = \frac{i}{2mw}(c(t) - imw(a(t) + d(t)) - m^2w^2\; b(t)) \qquad (33a)$$

$$\beta(t) = \frac{i}{2mw}\,(c(t) - imw(a(t) - d(t)) + m^2w^2\, b(t)) \qquad (33b)$$

$$\gamma\;(t) = \sqrt{\frac{mw}{2\hbar}}(y_1(t) + \;\frac{i}{mw}\; y_2(t)). \qquad (33c)$$

Similarly,

$$A_+^{\;+}(t) = U^+(t)\;A_+(t)\,U(t) \;\;=\;\; \beta\;^*(t)\;A + \alpha\;^*(t)\,A^+ \;+\; \gamma\;^* \qquad (34a)$$

which can also be derived from equation (32) by taking hermitian adjoints of the two sides.

Note that using equation (20 and 24b)

$$\alpha(0) = 1, \beta(0) = 0, \gamma(0) = 0 \qquad (34b)$$

For completeness, we give below the expression for $A_{-}(t)$ and $A_{--}^{+}(t)$

Indeed

$$A_{-}(t) = \alpha^{*}(t)\, A - \beta\,(t)\, A^{+} + \gamma'(t) \tag{35a}$$

$$A_{-}^{+}(t) = -\,\beta^{*}(t)\, A + \alpha\,(t)\, A^{+} + \gamma'^{*}(t) \tag{35b}$$

where

$$\gamma'(t) = \sqrt{\frac{mw}{2\hbar}}[(-d(t) + \frac{i}{mw}c(t)\; y_1(t) + (b(t) - \frac{i}{mw}\,a(t)\; y_2(t)]\ \text{-(36)}$$

As in equation (34b)

$$\gamma'(0) = 0$$

4. Recursion relations for the transition amplitudes $a_{mn}(t) = <m|\, U(t)\, |\, n>$.

Following the methods of our earlier paper [10]

$$a_{mn}(t) = <m|U(t)|n> = \frac{1}{\sqrt{n}} <m|U(t)A^{+}|n-1>$$

$$= \frac{1}{\sqrt{n}} <m|U(t)A^{+}U^{+}(t)U(t)|n-1>$$

$$= \frac{1}{\sqrt{n}} < A_{-}^{+}U(t)|n-1>$$

$$= \frac{1}{\sqrt{n}} <m|(-\beta^{*}(t)\,A + \alpha\,(t)\,A^{+} + \gamma'^{*}(t))U(t)|n-1)$$

using equation (35b).

$$= \frac{1}{\sqrt{n}}\,[<m| - \beta^{*}(t)U(t)(U^{+}(t)AU(t))|n-1> + \alpha\,(t)\,\sqrt{m} <m-1|U(t)|n-1>$$
$$+\gamma'^{*}(t), <m\,|U(t)|\,n-1>]$$

$$= \frac{1}{\sqrt{n}}[< m| - \beta^*(t)\, U(t)(\alpha\,(t) A + \beta\,(t) A^+ + \gamma\,(t)\,|n-1>$$

$$+ \alpha\,(t)\sqrt{m} < m-1|\,U(t)\,|n-1> + \gamma'^*\,(t) < m|U(t)|\,n-1>]$$

using equation (32)

$$= \frac{1}{\sqrt{n}}[-\alpha\,(t)\beta^*(t)\sqrt{n-1} < m|U(t)|n-2> - |\beta|^2\sqrt{n} < m|U(t)|n>$$

$$+ \alpha\,(t)\sqrt{m} < m-1|U(t)|n-1> + (-\beta^*(t)\gamma\,(t) + \gamma'^*(t) < m|U(t)|n-1>] \qquad (35b)$$

Transferring the term $|\beta|^2$ <m |U(t) | n> to the left hand side and noting that

$$|\alpha|^2 - |\beta|^2 = 1 \qquad -(39)$$

which can be observed directly from equations (33 a, b) and also from

$$[A_-(t)\,, A_-^+(t)] = 1 = |a|^2 - |\beta|^2$$

using equations (32, 33), we arrive at

$$< m|U(t)|n> = -\sqrt{\frac{n-1}{n}}\,\frac{\beta^*(t)}{\alpha^*(t)} < m|U(t)|n-2)$$

$$+ \sqrt{\frac{m}{n}}\,\frac{1}{\alpha^*(t)} < m-1|U(t)|n-1)$$

$$+ \frac{1}{\sqrt{n}}\,\frac{-\beta^*(t)\gamma\,(t) + \gamma'^*}{|\alpha|^2} < m|U(t)\;n-1>. \qquad -(40)$$

A similar procedure results in another recursion relation

$$< m\,|U(t)|n> = \sqrt{\frac{m-1}{m}}\,\frac{\beta\,(t)}{\alpha^*(t)} < m-2|U(t)|n>$$

$$+ \sqrt{\frac{n}{m}}\,\frac{1}{\alpha^*(t)} < m-1|U(t)|n-1>$$

$$+ \frac{1}{\sqrt{m}}\,\frac{\beta\,(t)\gamma'^*(t) + \gamma\,(t)}{|\alpha|^2} < m-1|U(t)|n> \qquad -(40b)$$

5. <u>Calculation of the transition amplitudes</u>

We define coefficients $B_{mn}(t)$ by means of the equation

$$a_{mn}(t) = < m|U(t)|n > = \sqrt{m!n!} \left(\frac{\beta(t)}{\alpha^*(t)}\right)^{m/2} \left(\frac{\beta^*(t)}{\alpha^*(t)}\right)^{n/2} B_{mn}(t) \qquad -(41)$$

In terms of $B_{mn}(t)$, the recursion relations in equation (32) become

$$n\ B_{mn}(t) = -\ B_{mn-2}(t) + \frac{1}{|\beta|} B_{m-1n-1}(t) + B(t)\ B_{mn-1}(t) \qquad -(42a)$$

$$m\ B_{mn}(t) = B_{m-2n}(t) + \frac{1}{|\beta\ |} B_{m-1n-1}(t) + A(t)\ B_{m-1n}(t) \qquad (42b)$$

where we have used

$$A(t) = \sqrt{\frac{\alpha^*(t)}{\beta\ (t)}}\ \frac{\beta\,(t)\gamma'^*(t) + \gamma\ (t)}{|\alpha\,|^2\,(t)} \qquad (43a)$$

$$B(t) = \sqrt{\frac{\alpha^*(t)}{\beta\ (t)}}\ \frac{-\beta\ ^*(t)\gamma\ (t) + \gamma'^*(t)}{|\alpha\,|^2\,(t)} \qquad (43b)$$

The recursion relations in equation (42) are sufficient to determine all $B_{mn}(t)$ and hence the transition amplitudes up to a (in general) complex constant. To determine $B_{mn}(t)$, we define a generating function.

$$G(x,y,t) = \sum_{m,n=0}^{\infty} B_{mn}(t) x^m y^n \qquad (44)$$

where we shall assume that any $B_{mn}(t)$ with any index taking negative integral value is zero. Then the recursion relations in equation (42) give

$$\frac{\partial}{\partial y} G(x,y,t) = (-y + \frac{1}{|\beta|(t)} x + B(t))\ G\,(x,y,t), \qquad -(45a)$$

$$\frac{\partial}{\partial x} G(x,y,t) = (\,x + \frac{1}{|\,\beta\,|(t)} y + A(t))\ \ G(x,y,t) \qquad (45b)$$

which can be easily solved to give

$$G(x,y,t) = G(0,0,t)\,\exp\left(\frac{x^2 - y^2}{2} + \frac{xy}{|\beta|} + A(t)\,x + B(t)\,y\right) \qquad -(46)$$

The t-dependent function G(0,0,t) is related to $a_{c0}(t)$.
Indeed $G(0,0,t) = B_{00}(t) = a_{00}(t) = \langle 0|\, U(t)\, |0\rangle$ (47)

Thus,

$$G(x,y,t) = a_{00}(t) \sum \frac{(-1)^{\ell} (\frac{1}{2})^{k+\ell}\; x^{2k+p+q}\; y^{2\ell+p+\gamma}}{k!\;\ell!\;p!\,q!\;r!\,|\beta|^{p}}\, A(t)^{q}\, B(t)^{\gamma} \qquad -(48)$$

To obtain $B_{mn}(t)$ which from equation (44) is the coefficient of $x^m\, y^n$ in the above expansion, we take

$$m = 2\,k + p + q, \quad n = 2l + p + r$$

which fix k and l as

$$k = \frac{m - p - q}{2}, \quad \ell = \frac{n - p - r}{2} \qquad -(49)$$

and results in

$$B_{mn}(t) = a_{00}(t) \sum_{p,q,r} (-1)^{\frac{n-p-r\gamma}{2}} \frac{\left(\frac{1}{2}\right)^{\frac{m+n-q-r\gamma}{2}-p}}{\left(\frac{m-p-q}{2}\right)!\left(\frac{n-p-r}{2}\right)!\,p!\,q!\,r!\,|\beta^{p}|}\, A(t)^{q}\, B(t)^{r}$$

where the summations over p, q, r are restricted by the argument of any factorial present in equation (50) to be a non-negative integer. For example, if m is an even integer, then p + q have to be even and $m - p - q \geq 0$. We note, in particular, that when $m = n = 0$, $p = q = \gamma = 0$ and $B_{00}(t) = a_{00}(t)$ which is consistent with equation (47).

From equations 4 and 50, we have

$$a_{mn}(t) = a_{00}(t)\,\sqrt{m!n!}\left(\frac{\beta\,(t)}{\alpha^{*}(t)}\right)^{m/2}\left(\frac{\beta^{*}(t)}{\alpha^{*}(t)}\right)^{n/2} *$$

$$*\sum_{p,q,r}(-1)^{\frac{n-p-r}{2}}\frac{\left(\frac{1}{2}\right)^{\frac{m+n-q-r\gamma}{2}-p} A(t)^{q}\;B(t)^{r}}{\left(\frac{m-p-q}{2}\right)!\left(\frac{n-p-r}{2}\right)!\,p!\,q!\,r!\,|\beta|^{\rho}} \tag{51}$$

which gives $a_{mn}(t)$ up to a function $a_{00}(t)$. To determine this function, we use the standard normalization

$$1 = <0|0> = \sum_{n=0}^{\infty} <0|U(t)|n> <n|U^{+}(t)|0> = \sum_{n=0}^{\infty} |a_{on}(t)|^{2}. \tag{52}$$

But from equation (51)

$$a_{0n}(t) = a_{00}(t)\,\sqrt{n}!\left(\frac{\beta^{*}(t)}{\alpha^{*}(t)}\right)^{n/2}\sum_{\gamma}\frac{(-1)^{\frac{m-r\gamma}{2}}\left(\frac{1}{2}\right)^{\frac{n-r}{2}} B^{*}(t)^{r}}{\left(\frac{n-r}{2}\right)!\,r!} \tag{53}$$

From the above equations,

$$|a_{00}(t)|^{2} = \left[\sum_{n\,t_1 t_2}\left(-\frac{1}{2}\right)^{n-\frac{t_1+t_2}{2}}\left(\frac{|\beta|}{|a|}\right)^{n}\frac{B(t)^{t_1}\;B^{*}(t)^{t_2}}{t_1!\,t_2!\left(\frac{n-t_1}{2}\right)!\left(\frac{n-t_2}{2}\right)!}\right]^{-1} \quad -(54)$$

where the expression in the bracket is obviously non-negative. Also the summation over n through infinite is convergent as $|\alpha| = \sqrt{1+|\beta|^{2}}$ *or* $0 < \frac{|\beta|}{|\alpha|} < 1.$

We can choose the phase such that $a_{oo}(t) = (\alpha(t) / \alpha(t)^{*})^{1/2} \,.\, | a_{00}(t) |$. This reduces to the case of linear harmonic oscillator: when f(t) = g(t) = 1, u(t) = v(t) = 0.
For the case when the linear terms are absent, A(t) = B(t) = 0 and the summations over q and r contain only the term with q = r = 0. Then p takes either odd or even integral values depending upon whether m and n are both odd or even. Thus in this case transitions from odd to even or from even to odd states are not allowed. However, in the presence of the linear terms, whether time-dependent or time-independent, transitions from odd or even or from even to odd states are now allowed as the corresponding $a_{mn}(t)$ may not vanish.

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