Photon gas as a classical medium

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Analogy between a photon in a plasma and a free material particle

$$\varepsilon = c\sqrt{p^2 + m_0^2 c^2}$$

or

$$\varepsilon = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Photon in a Vacuum

$$v \leqslant c$$

$$\omega = kc, \qquad v_\varphi = \frac{\omega}{k} = \frac{\partial \omega}{\partial k} = c.$$

$$\varepsilon_\gamma = p.c$$

If $p = 0,\ \varepsilon_\gamma = 0\ (\omega = 0)$

we can say in the vacuum photon exists only in motion. However the light can be stopped in different mediums

$$(\overrightarrow{p} = 0\ \overrightarrow{k} = 0 \text{ but } \omega \neq 0, \varepsilon_\gamma \neq 0\ )$$

For a plasma

$$\omega = \sqrt{\omega_p^2 + k^2c^2}$$

or

$$\varepsilon_\gamma = c\sqrt{p^2 + m_\gamma^2 c^2}$$

or

$$\varepsilon_\gamma = \frac{m_\gamma c^2}{\sqrt{1 - \frac{u_\gamma^2}{c^2}}}$$

$$m_\gamma = \frac{\hbar \omega_p}{c^2}$$

$$u_\gamma = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \leq c$$

$$u_\gamma = \frac{\partial \omega}{\partial k}$$ is the group velocity.

Momentum of the photon

$$\overrightarrow{p_\gamma} = \hbar \overrightarrow{k} = \frac{m_\gamma}{\sqrt{1 - \frac{u_\gamma^2}{c^2}}} . \overrightarrow{u_\gamma}$$

Thus the wave packets of light propagate with a group velocity $(u_\gamma < c)$ in accordance with the theory of relativity

Skin depth

$$\lambda_c = \frac{2\pi c}{\omega_p} = \frac{2\pi\hbar}{m_\gamma c}$$

takes the simple meaning of the Compton wavelength of a photon in a plasma

In the relativistic theory a coordinate uncertainty in a frame of reference in which the particle is moving with energy

$$\Delta q \sim \frac{c\hbar}{\varepsilon}$$

For Photons

$$\Delta q = \frac{c}{\omega} = \begin{cases} \lambda & \omega_p^2 << k^2c^2 \\ \lambda_c & \omega_p^2 >> k^2c^2 \end{cases}$$

or the characteristic dimensions of the problem should be large in comparison with the wavelength or the Compton length

In the quantum field theory the eigenvalues of the Hamiltonian are

$$E = \sum_{\overrightarrow{k},\sigma} \left(n_{\overrightarrow{k},\sigma} + \frac{1}{2} \right) \hbar\omega(\overline{k})$$

- These expressions enables to introduce the concept of photons, i.e. the EM field as an ensemble of particles each with energy $\eta\omega$ and momentum $\eta\vec{k}$

- The occupation numbers $n_{\vec{k},\sigma}$ now represent the number of photons with given $\vec{k}$ and Polarization σ

- The number of levels of the energy spectrum increase exponentially with the number of photons N and separation between levels are given by number of the 10^{-N}.

State Equation of EM Waves

$$\omega = \omega(k)$$

In the homogeneous and nondisipative plasma, we had already dispersion equations

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

here $\omega_p = const,\ k = const$

This is case, when amplitude of EM field is constant

$E(\overrightarrow{r}, t)$ **is the geometric approximation**

$$\frac{k^2(\overrightarrow{r}, t)c^2}{\omega^2} = 1 - \frac{\omega_p^2(\overrightarrow{r}, t)}{\gamma(\overrightarrow{r}, t).\omega^2}$$

Wigner-Moyal Equation in quantum theory

$$\left[\frac{\partial}{\partial t}+\frac{\overrightarrow{p}}{m}\frac{\partial}{\partial\overline{r}}-\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\frac{\partial}{\partial\overline{p}}.\frac{\partial}{\partial\overline{r}}\right)\right]V(\overline{r})F(\overline{r},\overline{p},t)=0$$

Relativistic Kinetic Equation for the Photon Gas

For the photon gas

$$\frac{\partial}{\partial t}N(\overline{k},\omega,\overline{r},t)+\frac{c^2}{\omega}(\overline{k}.\overline{\nabla})N(\overline{k},\omega,\overline{r},t)$$

$$-\omega_p^2\sin\frac{1}{2}(\nabla_r.\nabla_{\overline{k}}-\frac{\partial}{\partial t}\frac{\partial}{\omega})\rho\frac{N}{\omega}=0$$

where $\rho=\frac{n_e}{n_{0e}}\frac{1}{\gamma}$, γ **is the relativistic gamma factor of the electrons** gamma can be expressed as

$$\gamma(\overline{r},t)=\sqrt{1+Q}=\sqrt{1+\beta\int\frac{d\overline{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}\frac{N(\overline{k},\omega,\overline{r},t)}{\omega}}$$

$$\beta=\frac{2\hbar\omega_p^2}{m_0 n_0 c^2}$$

The total <u>number of Photons</u>

$$N=2\int d\overline{r}\int\frac{d\overline{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}N(\overline{k},\omega,\overline{r},t)=const$$

Hence the chemical potential of the photon gas is not zero

In the geometric optics approximation the one-particle Liouville-Vlasov equation with an additional term

$$\frac{\partial N}{\partial t} + \frac{c^2}{\omega}(\overline{k}.\overline{\nabla})N - \frac{\omega_p^2}{2}(\nabla\rho.\nabla_k - \frac{\partial\rho}{\partial t}\frac{\partial}{\omega})\frac{N}{\omega} = 0$$

here there are two forces of direct nature which can change the occupation number of photons

$$\nabla\rho = \nabla\left(\frac{n_e}{n_a}\right) = \frac{1}{\gamma}\left(\nabla\frac{n_e}{n_a} - \frac{1}{\gamma}\frac{n_e}{n_a}\nabla\gamma\right)$$

1st one is just Compton scattering process, 2nd is new type of Compton scattering photon scatter on the wave packet

Existence of the longitudinal photons

We have shown that $\frac{\delta n_e}{n_{0e}} << \frac{\delta n_\gamma}{n_{0\gamma}}$

$$1+\frac{\omega_p^2}{2\gamma^3}\beta\int\frac{d^3k}{\omega(k)}\left\{\frac{N_0^\perp(\overline{k}+\overline{q}/2)}{\omega(\overline{k}+\overline{q}/2)}-\frac{N_0^\perp(\overline{k}-\overline{q}/2)}{\omega(\overline{k}-\overline{q}/2)}\right\}\frac{1}{\frac{\overline{q}\overline{k}c^2}{\omega(k)}-\Omega}=0$$

Ω and $\overline{q}$ **are the frequency and wave vector of the longitudinal photons, well known Bogoliubov energy spectrum**

$$\varepsilon(p)=\sqrt{v^2.p^2+\left(\frac{p^2}{2m_{eff}}\right)^2}$$

who developed microscopic theory of the super fluidity

Adiabatic Photon self-capture

γ **is a slowly variation function in space and time**

Assuming inequality $t > L/u_g$ **& neglecting the time derivatives in the kinetic equation, we obtain**

$$N\left(\bar{k}, \omega, \bar{r}, t, \omega(k)\right) = Ae^{-\frac{k^2+k_p^2\delta\rho}{2\delta_k^2}}$$

Where $k_p^2 = \frac{\omega_p^2}{c^2}$

and

$$\delta\rho = \frac{\delta n_e}{n_0}\frac{1}{\gamma} + \frac{n_e}{n_0}\delta\left(\frac{1}{\gamma}\right) \simeq \delta\left(\frac{1}{\gamma}\right) = \frac{1}{\gamma} - \frac{1}{\gamma_0}$$

$$\frac{n_\gamma^{trap}}{n_{0\gamma}} = \frac{4}{3\sqrt{\pi}}\left(\frac{k_p}{\sqrt{2}\sigma_0}\right)^3 |\delta\rho|^{3/2}$$

Pauli Equation for the Photon Gas

$$\frac{\partial N(\overline{k},t)}{\partial t} = \sum_{\pm} \int \frac{d^3q}{(2\pi)^3} \left[W \pm (\overline{k}+\overline{q},\overline{k}) N(\overline{k}+\overline{q},t) - W \pm (\overline{k},\overline{k}+\overline{q}) N(\overline{k},t) \right]$$

where $W \pm (\overline{k},\overline{q})$ is the scattering rate

Bose-Einstein Condensation in Photon Gas

Some times the great scientist can make mistakes

$$n_p = \frac{1}{\exp\left(\frac{\hbar\omega}{T}\right) - 1}$$

$$\overline{k} = 0, \; \omega = 0, \; n_n \to \infty$$

$$N(\overline{p}, \overline{r}) = \frac{1}{\exp\left(\frac{\varepsilon_k - \mu + m_\gamma c^2}{T_\gamma}\right) - 1}$$

where $\varepsilon_k = c\sqrt{p^2 + m_\gamma^2 c^2} - m_\gamma c^2$

The Photon density

Because $n_\gamma > 0$

$$n_\gamma = 2\int \frac{d\overline{p}}{(2\pi\hbar)^3} N(\overline{p}, r)$$

in any point of space $m_\gamma(\overline{r})c^2 > \mu_\gamma$

The critical temperature of the B.E condensation is determined for the fixed points

$$m_\gamma(r_f)c^2 = \frac{\hbar\omega_p}{\gamma^{1/2}(r_f)} = \mu_\gamma$$

this condition, can be define the critical temperature For non-relativistic temperature

$$T_c \sim n^{2/3}$$

For ultra relativistic case

$$T_c \sim n^{1/3}$$

When the temperature is below critical T_c chemical potential becomes zero, and occupation number would be in the form

$$N(\overline{p},\overline{r}) = \frac{1}{\exp\left(\frac{\varepsilon_k}{T_\gamma}\right) - 1} + 4\pi^3 n_{0\gamma}\delta(\overline{p})$$

The problem of BEC and evaporation of the Bose-Einstein condensate can be investigated by Fokker-Planck equation, which we shall derive Using Pauli equation, We suppose that

$$|\overline{q}| << |\overline{k}|, \quad \text{and } \Omega << \omega$$

$$W(\overline{k}+\overline{q},\overline{k})N(\overline{k}+\overline{q}) \approx W(k)N(k) + \overline{q}\frac{\partial}{\partial \overline{k}}(WN)_{q=0} + \frac{q_i q_j}{2}\frac{\partial^2 WN}{\partial k_i \partial k_j}$$

$$\frac{\partial N}{\partial t} = a\frac{\partial}{\partial \bar{k}}(\bar{k}N) + \frac{D_0}{2}\nabla_k^2 N$$

where $a = \frac{D_0}{2\sigma_k^2}$ and D_0 **are the dynamic friction and diffusion coefficients, respectively**

First we neglect the diffusion term and consider 1D case, the solution of which is

$$N = \frac{f}{k} - cont\ e^{at}$$

Second

$$\frac{\partial N}{\partial t} = \frac{D_0}{2}\nabla_k^2 N$$

Assuming that initially all the photons are in ground sate with

$$k = 0, \text{ or } N_0 = 4\pi^3 n_0 \delta(\bar{k})$$

The solution is

$$N(k,t) = \frac{n_0 e^{-\frac{k^2}{2D_0 t}}}{(2\pi D_0 t)^{1/2}}$$

From here

$$< k^2 >= D_0 t$$

We have derived a relation between the diffusion time, t_D and the time of condensation

$$t_p / t_c = k^2 r_0^2, \text{ which is always} >> 1$$