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S-Dualities in noncommutative and nonanticommutative field theories

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PLAN

- Introduction and Summary.
- Dual action for non–anticommuting $U(1)$ theory.
- Equivalence of partition functions.

Introduction and Summary

Parent action ($g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$)

$$S_m = \int d^4x(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\mu A_D^\nu F^{\rho\sigma}). \tag{1}$$

Treat F as an independent variable without requiring any relation with the gauge field A. Performing path integral over A_D, which is equivalent to solve the equations of motion for A_D in terms of F and replace it in the action, leads to Abelian gauge theory action

$$S_o = -\frac{1}{4g^2}\int d^4x F_{\mu\nu}F^{\mu\nu}, \tag{2}$$

now $F = dA$. Performing path integral over F, which is equivalent to solve the equations of motion for F in terms of A_D and replace it in the action, leads to the dual action

$$S_D = -\frac{g^2}{4}\int d^4x {F_D}^{\mu\nu}F_{D\mu\nu} \tag{3}$$

where $F_D = dA_D$.

Duality transformation is

$$g \to \frac{1}{g}.$$

Noncommutative space is defined by constant θ :

$$[x^\mu, x^\nu] = \theta^{\mu\nu}$$

Duality can be generalized to noncommutative gauge theory:

$$g \to \frac{1}{g}.$$

and

$$\theta^{\mu\nu} \to \tilde{\theta}^{\mu\nu} = g^2 \epsilon^{\mu\nu\rho\sigma} \theta_{\rho\sigma}.$$

yielding noncommutativity of space–time coordinates of the dual one although the original noncommutativity was between space–space coordinates.

- It is shown that noncommutativity of space–time coordinates resulting from duality does not require any change in Hamiltonian formalism. Thus we derived Hamiltonian formulation of noncommutative $D3$–branes and studied some aspects.

- For supersymmetric $U(1)$ theory parent action was available in terms of restricted superfields. A version in component fields is introduced.

- Seiberg–Witten map between ordinary and noncommutative gauge fields is generalized to supersymmetric gauge theories.

- Different parent actions, thus different dual actions for noncommuting supersymmetric $U(1)$ theory are obtained.

- Dual actions for noncommuting theories were given by solving eqs. of motion. Constrained Hamiltonian methods are used to show that partition functions of them are equivalent.

- *Parent action for nonanticommuting* $U(1)$ *gauge theory is introduced and dual action is derived. Equivalence of the partition functions is shown.*

Dual action for non–anticommuting $U(1)$ theory.

Superstring theory with pure spinors in a graviphoton background gives rise to a non-anticommutative superspace. It can equivalently be introduced as deformation of 4 dimensional $N = 1$ superspace by making the chiral fermionic coordinates θ_α, $\alpha = 1, 2$, non–anticommuting

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta},$$

where $C^{\alpha\beta}$ $\left(C^{\mu\nu} = C^{\alpha\beta}\epsilon_{\beta\gamma}\sigma^{\mu\nu}_{\alpha}{}^{\gamma}\right)$ are constant deformation parameters. $\bar{\theta}_\alpha$ are intact.

This breaks half of the supersymmetry.

Moyal antibrackets (star products) are used. Thus, instead of operators, one deals with the usual superspace variables.

In euclidean $\mathbb{R}^4$ chiral and antichiral fermions are not related with complex conjugation. Seiberg used the vector superfield of this deformed superspace to derive, after a change of variables like Seiberg–Witten map, the $N = \frac{1}{2}$ supersymmetric Yang-Mills theory action

$$I_{1/2} = \frac{1}{g^2}\int d^4x Tr\Big\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} - i\lambda\not{D}\bar{\lambda} + \frac{1}{2}D^2 - \frac{i}{2}C^{\mu\nu}G_{\mu\nu}(\bar{\lambda}\bar{\lambda}) + \frac{|C|^2}{8}(\bar{\lambda}\bar{\lambda})^2$$

Gauge transformations possess the usual form. Although we deal with euclidean $\mathbb{R}^4$, we use Minkowski space notation.

We propose the parent action as

$$I_p = I_0[\mathrm{X}] + I_l[\mathrm{X}, \mathrm{X}_D]$$

$$\begin{aligned} I_0 &= \frac{1}{g^2}\int d^4x\Big\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\lambda\partial\!\!\!/\bar\lambda - \frac{i}{2}\bar\psi\bar{\partial\!\!\!/}\psi + \frac{1}{4}D_1^2 + \frac{1}{4}D_2^2 \\ &\quad -\frac{i}{4}C^{\mu\nu}F_{\mu\nu}(\bar\lambda\bar\lambda + \bar\psi\bar\psi)\Big\} \end{aligned}$$

$$\begin{aligned} I_l &= \int d^4x\Big\{\frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}\partial_\lambda A_{D\kappa} + \frac{1}{2}\lambda\partial\!\!\!/\bar\lambda_D + \frac{1}{2}\lambda_D\partial\!\!\!/\bar\lambda - \frac{1}{2}\bar\psi\bar{\partial\!\!\!/}\lambda_D \\ &\quad -\frac{1}{2}\bar\lambda_D\bar{\partial\!\!\!/}\psi + \frac{i}{2}D_D(D_1 - D_2)\Big\}. \end{aligned}$$

Here $F_{\mu\nu}$ are independent field variables.

The equations of motion with respect to the "dual" fields are

$$\epsilon^{\mu\nu\lambda\kappa}\partial_\nu F_{\lambda\kappa} = 0\ ,$$

$$\partial\!\!\!/\bar\psi = \partial\!\!\!/\bar\lambda \quad , \quad \bar{\partial\!\!\!/}\psi = \bar{\partial\!\!\!/}\lambda, \quad , \quad D_1 = D_2 = D\ .$$

One solves by setting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. When one plugs this and the solutions of the other equations of motion in terms of λ, $\bar\lambda$, D, into the parent action, the non–anticommuting $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory action follows:

$$I = \frac{1}{g^2}\int d^4x\Big\{-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - i\lambda\partial\!\!\!/\bar\lambda + \frac{1}{2}D^2 - \frac{i}{2}C^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)\bar\lambda\bar\lambda\Big\}.$$

The equations of motion with respect to the "original" fields are

$$\frac{1}{2g^2}F^{\mu\nu}+\frac{i}{4g^2}C^{\mu\nu}(\bar\lambda\bar\lambda+\bar\psi\bar\psi)-\frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}\partial_\lambda A_{D_\kappa}=0\ ,$$
$$\partial\!\!\!/\bar\lambda+ig^2\partial\!\!\!/\bar\lambda_D=0\quad,\quad\partial\!\!\!/\bar\psi-ig^2\partial\!\!\!/\bar\lambda_D=0\ ,$$
$$\bar\partial\!\!\!/\lambda+C^{\mu\nu}F_{\mu\nu}\bar\lambda+ig^2\bar\partial\!\!\!/\lambda_D=0\quad,\quad\bar\partial\!\!\!/\psi+C^{\mu\nu}F_{\mu\nu}\bar\psi-ig^2\bar\partial\!\!\!/\lambda_D=0\ ,$$
$$D_1+ig^2D_D=0\quad,\quad D_2-ig^2D_D=0\ .$$

We solve the equations of motion for the dual fields and substitute them in the parent action to obtain the dual theory action :

$$I_D=g^2\int d^4x\Big\{-\frac{1}{4}F_D^{\mu\nu}F_{D\mu\nu}-i\lambda_D\partial\!\!\!/\bar\lambda_D+\frac{1}{2}D_D^2+\frac{i}{4}g^2\epsilon^{\mu\nu\lambda\kappa}C_{\mu\nu}F_{D\lambda\kappa}\bar\lambda_D\bar\lambda_D\Big\},$$

where $F_{D\mu\nu}=\partial_\mu A_{D\nu}-\partial_\nu A_{D\mu}$.

Observe that the original theory action and its dual possess the same form and the duality transformation is

$$g\ \rightarrow\ \frac{1}{g}$$
$$C^{\mu\nu}\ \rightarrow\ C_D^{\mu\nu}=-\frac{1}{2}g^2\epsilon^{\mu\nu\lambda\kappa}C_{\lambda\kappa}=ig^2C^{\mu\nu}.$$

Equivalence of partition functions

Partition function for the parent action is expected to produce partition functions of the daughter actions. There are some terms cubic in fields. Thus, it would be apposite to discuss its partition function in phase space, where integrations would be simplified due to hamiltonian constraints.

$$(P^{\mu\nu},\ \Pi_1^{\alpha},\ \bar{\Pi}_{1\dot{\alpha}},\ \Pi_2^{\alpha},\ \bar{\Pi}_{2\dot{\alpha}},\ P_1,\ P_2); (P_D^{\mu},\ \Pi_D^{\alpha},\ \bar{\Pi}_{D\dot{\alpha}},\ P_D)$$

are the canonical momenta corresponding to

$$(F_{\mu\nu},\ \lambda_\alpha,\ \bar{\lambda}^{\dot{\alpha}},\ \psi_\alpha,\ \bar{\psi}^{\dot{\alpha}},\ D_1,\ D_2);\ (A_{D\mu},\ \lambda_{D\alpha},\ \bar{\lambda}_D^{\dot{\alpha}},\ D_D).$$

Each of the canonical momenta resulting from the parent action gives rise to a primary constraint, which we collectively denote them as $\{\Theta^a\}$:

$$\begin{aligned}
\phi_1^{0i} &\equiv P^{0i} \approx 0, & \phi_2^{ij} &\equiv P^{ij} \approx 0 ,\\
\chi_1^{\alpha} &\equiv \Pi_1^{\alpha} \approx 0, & \bar{\chi}_{1\dot{\alpha}} &\equiv \bar{\Pi}_{1\dot{\alpha}} - \tfrac{i}{2g^2}\lambda^{\alpha}\sigma^0_{\alpha\dot{\alpha}} + \tfrac{1}{2}\lambda_D^{\alpha}\sigma^0_{\alpha\dot{\alpha}} \approx 0,\\
\chi_{2\dot{\alpha}} &\equiv \bar{\Pi}_{2\dot{\alpha}} \approx 0, & \chi_2^{\alpha} &\equiv \Pi_2^{\alpha} - \tfrac{i}{2g^2}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\alpha} - \tfrac{1}{2}\bar{\lambda}_{D\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\alpha} \approx 0,\\
\Phi_1 &\equiv P_1 \approx 0, & \Phi_2 &\equiv P_2 \approx 0,\\
\phi_{D_1} &\equiv P_D^0 \approx 0, & \phi_{D_2}^i &\equiv P_D^i - \tfrac{1}{2}\epsilon^{ijk}F_{jk} \approx 0,\\
\chi_D^{\alpha} &\equiv \Pi_D^{\alpha} - \frac{1}{2}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\alpha} \approx 0, & \chi_{D\dot{\alpha}} &\equiv \bar{\Pi}_{D\dot{\alpha}} + \tfrac{1}{2}\lambda^{\alpha}\sigma^0_{\alpha\dot{\alpha}} \approx 0,\\
\Phi_D &\equiv P_D \approx 0.
\end{aligned}$$

Canonical hamiltonian associated with the parent action is

$$
\begin{aligned}
\mathcal{H}_p &= \frac{1}{g^2}[\frac{1}{4}F_{\mu\nu}^2 + \frac{i}{2}\lambda\bar{\nabla}\bar{\lambda} + \frac{i}{2}\bar{\psi}\bar{\nabla}\psi - \frac{1}{4}(D_1^2 + D_2^2) + \frac{i}{4}C^{\mu\nu}F_{\mu\nu}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi})] \\
&\quad -\epsilon^{ijk}F_{0i}\partial_j A_{D\kappa} + \frac{1}{2}\epsilon^{ijk}F_{ij}\partial_\kappa A_{D0} - \frac{1}{2}\lambda\bar{\nabla}\bar{\lambda}_D - \frac{1}{2}\lambda_D\bar{\nabla}\bar{\lambda} + \frac{1}{2}\bar{\psi}\bar{\nabla}\lambda_D \\
&\quad +\frac{1}{2}\bar{\lambda}_D\bar{\nabla}\psi - \frac{i}{2}D_D(D_1 - D_2).
\end{aligned}
$$

Extended hamiltonian is obtained by adding the primary constraints Θ^a with the help of Lagrange multipliers l_a, to the canonical hamiltonian:

$$\mathcal{H}_E = \mathcal{H}_p + l_a\Theta^a$$

Consistency of the primary constraints with the equations of motion:

$$\dot{\Theta}^a = \{\mathcal{H}_E, \Theta^a\} \approx 0$$

gives rise to the secondary constraints

$$
\begin{aligned}
\Delta_1 &\equiv \{\mathcal{H}_p, P_1\} = -\frac{1}{2g^2}D_1 - \frac{i}{2}D_D \approx 0 \, , \\
\Delta_2 &\equiv \{\mathcal{H}_p, P_2\} = -\frac{1}{2g^2}D_2 + \frac{i}{2}D_D \approx 0 \, , \\
\Delta_D &\equiv \{\mathcal{H}_p, P_D\} = \frac{i}{2}(D_1 - D_2) \approx 0 \, , \\
\varphi_D &\equiv \{\mathcal{H}_p, P_D^0\} = \frac{1}{2}\epsilon^{ijk}\partial_k F_{ij} \approx 0 \, , \\
\varphi_1^{0i} &\equiv \{\mathcal{H}_p, P_{0i}\} = F^{0i} - g^2\epsilon^{ijk}\partial_j A_{Dk} + \frac{ig^2}{2}C^{0i}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \approx 0 \, .
\end{aligned}
$$

In path integrals first and second class constraints are treated on different grounds. Thus, let us first identify the first class constraints:

ϕ_{D1} is obviously first class. Moreover, we observe that the linear combination

$$\phi_{D_3} \equiv \partial_i \phi^i_{D_2} + \varphi_D = \partial_i P^i_D \approx 0,$$

is also a first class constraint. There are no other first class constraints. However, the constraints $\phi^i_{D_2}$ contain second class constraints which we should separate out. This is due to the fact that a vector can be completely described by giving its divergence and rotation (up to a boundary condition). We used divergence of $\phi^i_{D_2}$, so that, there are still two linearly independent second class constraints following from the curl of $\phi^i_{D_2}$:

$$\phi^n_{D_4} \equiv K^n_i \phi^i_{D_2} = \mathcal{K}^{ni} \epsilon_{ijk} \partial^j \phi^k_{D_2} \approx 0,$$

where $n = 1,2$. $\mathcal{K}^n_i$ are some constants whose explicit forms are not needed for the purposes of this work. Although all of them are second class, we would like to separate φ^{0i}_1 in a similar manner:

$$\begin{aligned} \varphi_2 &\equiv \partial_i \varphi^{0i}_1 = -\partial_i F^{0i} - \frac{i}{2} C^{0i} \partial_i (\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \approx 0, \\ \varphi^n_3 &\equiv L^n_i \varphi^{0i}_1 = \mathcal{L}^{ni} \epsilon_{ijk} \partial^j \varphi^{0k}_1 \approx 0. \end{aligned} \tag{4}$$

where $\mathcal{L}^{nj}$ are some constants. The reason of preferring this set of constraints will be clear when we perform the path integrals, though explicit forms of $\mathcal{L}^n_i$ play no role in our calculations.

In phase space, partition function can be written as

$$\mathcal{Z} = \int \prod_i \mathcal{D}\mathrm{Y}_i\, \mathcal{D}P_{\mathrm{Y}_i}\, \mathcal{M}\, e^{i\int d^3x(\dot{\mathrm{Y}}_i P_{\mathrm{Y}_i} - \mathcal{H}_p)}$$

$$\mathcal{M} = N\det(\partial_i^2)\delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)\delta(P_{D0})\delta(A_{D0})\mathrm{sdet}\, M \prod_z \delta(S_z),$$

where Y_i and P_{Y_i} embrace all of the fields and their momenta. S_z denotes all second class constraints: $S_z \equiv (\phi_1,\, \phi_2,\, \Phi_1,\, \Phi_2,\, \phi_{D_4},\, \Phi_D,\, \varphi_2,\, \varphi_3,\, \Delta_1,\, \Delta_2,\, \varphi_D,\, \Delta_D,\, \chi_1,\, \bar{\chi}_1,\, \chi_2,\, \bar{\chi}_2,\, \chi_D,\, \bar{\chi}_D)$. We adopted the gauge fixing (auxiliary) conditions

$$A_{D_0} = 0\,,$$
$$\partial_i A_{D_i} = 0,$$

for the first class constraints ϕ_{D_1} and ϕ_{D_3}. N is a normalization constant. The matrix of the generalized Poisson brackets of the second class constraints $M = \{S_z, S_{z'}\}$ can be written in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

so that, its superdeterminant is given by

$$\mathrm{sdet}M = (\det D)^{-1}\det(A - BD^{-1}C).$$

Calculations of B, C and D can be shown to yield

$$(BD^{-1}C) = 0.$$

Therefore,

$$\mathrm{sdet} M = \frac{\det A}{\det D}.$$

Contribution of fermionic constraints is

$$\det D^{-1} = -(4 \det g^2)^2.$$

Here, $\det g^2$, which arise because we deal with constraints of a field theory, should appropriately be regularized. Contribution of the bosonic constraints has already been calculated:

$$\det A = \det \left(\epsilon_{ijk} \partial^i K_1^j K_2^k \right) \det \left(\epsilon_{ijk} \partial^i L_1^j L_2^k \right).$$

These determinants which are multiplication of three linear operators should be interpreted as multiplication of their eigenvalues.

The integrals over all of the fermionic momenta and $P_{\mu\nu}$ can be easily performed utilizing the related delta functions, to get

$$\begin{aligned}
\mathcal{Z} = & \int \mathcal{D}(\mathrm{All}) \tilde{\mathcal{M}} \exp\Big\{ i \int d^3x \Big[P_1 \dot{D}_1 + P_2 \dot{D}_2 + P_D^0 \dot{A}_{D0} + P_D^i \dot{A}_{Di} \\
& + P_D \dot{D}_D - \frac{1}{4g^2} F^{0i} F_{0i} - \frac{1}{4g^2} F^{ij} F_{ij} - \frac{i}{2g^2} \lambda \partial\!\!\!/ \bar{\lambda} \\
& - \frac{i}{2g^2} \bar{\psi} \bar{\partial\!\!\!/} \psi + \frac{1}{4g^2} (D_1^2 + D_2^2) - \frac{i}{2g^2} C^{0i} F_{0i} (\bar{\lambda} \bar{\lambda} + \bar{\psi} \bar{\psi}) \\
& - \frac{i}{4g^2} C^{ij} F_{ij} (\bar{\lambda} \bar{\lambda} + \bar{\psi} \bar{\psi}) + \epsilon^{ijk} F_{0i} \partial_j A_{Dk} - \frac{1}{2} \epsilon^{ijk} F_{ij} \partial_k A_{D0} \\
& + \frac{1}{2} \lambda \partial\!\!\!/ \bar{\lambda}_D + \frac{1}{2} \lambda_D \partial\!\!\!/ \bar{\lambda} - \frac{1}{2} \bar{\psi} \bar{\partial\!\!\!/} \lambda_D - \frac{1}{2} \bar{\lambda}_D \bar{\partial\!\!\!/} \psi + \frac{i}{2} D_D (D_1 - D_2) \Big] \Big\}.
\end{aligned}$$

Here, $\tilde{\mathcal{M}}$ is the same with $\mathcal{M}$ except the delta functions which we utilized above.

We first would like to integrate over the fields which do not carry the label "D" : P_1, P_2 integrals are trivially performed and by integrating over D_1 and D_2 we get a factor of $\det g^2$ and $\delta(D_D)$. Integrations over ψ and λ yield

$$(\det\partial\!\!\!/ / \det g^2)^2 \delta(i\bar{\psi} + g^2\bar{\lambda}_D)\delta(i\bar{\lambda} - g^2\bar{\lambda}_D).$$

Thus, we replace $\bar{\psi}$ with $ig^2\bar{\lambda}_D$ and $\bar{\lambda}$ with $-ig^2\bar{\lambda}_D$ after integrating over $\bar{\psi}$ and $\bar{\lambda}$. Integrations over $F^{\mu\nu}$ yield substitution of F^{0i} with

$$g^2\epsilon^{ijk}\partial_j A_{Dk} + \frac{i}{2}g^4 C^{0i}\bar{\lambda}_D\bar{\lambda}_D,$$

F^{ij} with $\epsilon^{ijk}P_{Dk}$ and cancellation of the determinant. Moreover, we integrate over A^0_D, P^0_D and choose the normalization constant N such that we get

$$\begin{aligned}
\mathcal{Z} \;=\; & \int \mathcal{D}A_{Di}\,\mathcal{D}\bar{\lambda}_D\,\mathcal{D}P_{Di}\,\mathcal{D}D_D\,\mathcal{D}P_D\,(\det g^2)\det\partial_i^2)(\det\partial\!\!\!/)^2 \\
& \delta(D_D)\delta(P_D)\,\delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D) \\
& \exp\Big\{i\int d^3x\Big[P^i_D\dot{A}_{Di} + P_D\dot{D}_D - \frac{1}{2g^2}P_{Di}P^i_D - iC^{0i}_D P_{Di}\bar{\lambda}_D\bar{\lambda}_D \\
& -\frac{g^2}{4}F^{ij}_D F_{Dij} - \frac{i}{2}g^2 C^{ij}_D F_{Dij}\bar{\lambda}_D\bar{\lambda}_D - ig^2\lambda_D\partial\!\!\!/\bar{\lambda}_D + \frac{g^2}{2}D^2_D\Big]\Big\}.
\end{aligned}$$

In the exponent we distinguish the first order lagrangian of the dual theory where Π^α_D and $\bar{\Pi}_{D\dot{\alpha}}$ are eliminated from the path integral by performing their integrations.

Now, let us integrate over the fields carrying the label "D": P_D integral is trivial. Integration over D_D contributes as

$$(\det g^2)\delta(D_1 + D_2)\delta(D_1 - D_2).$$

Integrations of the fermionic variables λ_D and $\bar{\lambda}_D$ lead to

$$\delta(-\partial\!\!\!/\bar{\psi} + \bar{\partial\!\!\!/}\psi)\delta(\bar{\partial\!\!\!/}\lambda - \bar{\partial\!\!\!/}\psi).$$

Due to the constraint $\varphi_D = 0$ we set

$$F_{ij} = \partial_i A_j - \partial_j A_i.$$

However, this replacement does not diminish the relevant number of physical phase space variables as it should be the case if the second class constraint φ_D has been taken properly into account. Therefore, we adopt the change of variables with the replacement

$$\begin{aligned} &\mathcal{D}F_{ij}\delta(\epsilon^{klm}\partial_k F_{lm})\delta(K_n^i(P_{Di} + \tfrac{1}{2}\epsilon_{ijk}F^{jk})) \to \\ &\to \det(\partial^2)\mathcal{D}A_i\delta(\partial_j A^j)\delta\left(K_n^i(P_{Di} + \epsilon_{ijk}\partial^j A^k)\right). \end{aligned}$$

Expressing A_{Di} and P_{Di} in terms of the fields $(A_i,\ F_{0i})$ by making use of the delta functions $\delta(K_i^n\phi_D^i)\delta(L_i^n\phi_1^{0i})\delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)$ contributes to the measure with

$$\left[(\det g^2)^2 \det(\partial^2) \det\left(\epsilon_{ijk}\partial^i K_1^j K_2^k\right) \det\left(\epsilon_{ijk}\partial^i L_1^j L_2^k\right)\right]^{-1}.$$

Hence, integrations over A_{Di} and P_{Di} can be performed to obtain

$$
\begin{aligned}
\mathcal{Z} \;=\; & \int \mathcal{D}A_i \, \mathcal{D}F_{0i} \mathcal{D}\lambda \, \mathcal{D}\bar{\lambda} \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}D_1 \, \mathcal{D}P_1 \, \mathcal{D}D_2 \, \mathcal{D}P_2 \\
& (\det g^2) \det(\partial_i^2) \, \delta(\partial \cdot \mathbf{A}) \delta(D_1 + D_2) \delta(D_1 - D_2) \, \delta(-\partial\!\!\!/ \bar{\psi} + \partial\!\!\!/ \bar{\lambda}) \\
& \delta(\bar{\partial\!\!\!/} \lambda - \bar{\partial\!\!\!/} \psi) \, \delta\left(\partial_i F^{0i} + \frac{i}{2} \partial_i C^{0i} (\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \right) \\
& \exp\Big\{ i \int d^3x \Big[\frac{1}{g^2} \left(F^{0i} + \frac{i}{4} C^{0i} (\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \right) \dot{A}_i \\
& + \dot{D}_1 P_1 + \dot{D}_2 P_2 - \frac{1}{2g^2} F^{0i} F_{0i} \\
& - \frac{1}{4g^2} (\partial_i A_j - \partial_j A_i)^2 - \frac{i}{2g^2} \lambda \partial\!\!\!/ \bar{\lambda} - \frac{i}{2g^2} \bar{\psi} \bar{\partial\!\!\!/} \psi + \frac{1}{4g^2} (D_1^2 + D_2^2) \\
& - \frac{i}{2g^2} C^{0i} F_{0i} (\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) - \frac{i}{4g^2} C^{ij} (\partial_i A_j - \partial_j A_i)(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \Big] \Big\}
\end{aligned}
$$

Integrating over D_2, P_2, ψ, $\bar{\psi}$ and renaming $D_1 = D$ and $P_1 = P$ yield

$$
\begin{aligned}
\mathcal{Z} \;=\; & \int \mathcal{D}A_i \, \mathcal{D}F_{0i} \mathcal{D}\lambda \, \mathcal{D}\bar{\lambda} \, \mathcal{D}D \, \mathcal{D}P \\
& (\det g^2)(\det \partial_i^2)(\det \partial\!\!\!/)^2 \delta(P) \delta(D) \delta(\partial \cdot \mathbf{A}) \\
& \delta\left(\partial_i F^{0i} + i \partial_i C^{0i} \bar{\lambda}\bar{\lambda} \right) \exp\Big\{ i \int d^3 \Big[\frac{1}{g^2} \left(F^{0i} + i C^{0i} \bar{\lambda}\bar{\lambda} \right) \dot{A}_i + \dot{D} P \\
& - \frac{1}{2g^2} F^{0i} F_{0i} - \frac{1}{4g^2} (\partial_i A_j - \partial_j A_i)^2 - \frac{1}{g^2} \lambda \partial\!\!\!/ \bar{\lambda} + \frac{1}{2g^2} D^2 \\
& - \frac{i}{g^2} C^{ij} (\partial_i A_j - \partial_j A_i) \bar{\lambda}\bar{\lambda} \Big] \Big\}.
\end{aligned}
$$

In terms of the change of variables

$$
\begin{aligned}
g^2 P^i &= F^{0i} + C^{0i} \bar{\lambda}\bar{\lambda}, \\
\mathcal{D}F^{0i} &= (\det g^2) \mathcal{D}P^i,
\end{aligned}
$$

we write the partition function as

$$\begin{aligned}\mathcal{Z} \;=\; &\int \mathcal{D}A_i\, \mathcal{D}P^i\, \mathcal{D}\lambda\, \mathcal{D}\bar{\lambda}\, \mathcal{D}D\, \mathcal{D}P \\ &(\det g^2)\, (\det \partial_i^2)(\det \not{\partial})^2\, \delta(D)\delta(P)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A}) \\ &\exp\Big\{ i \int d^3x \Big[P^i \dot{A}_i + \dot{D}_1 P_1 - \frac{g^2}{2}(P_i)^2 - iC^{0i}P_i\bar{\lambda}\bar{\lambda} \\ &-\frac{1}{4g^2}(\partial_i A_j - \partial_j A_i)^2 - \frac{i}{g^2}\lambda\not{\partial}\bar{\lambda} + \frac{1}{2g^2}D^2 \\ &-\frac{i}{2g^2}C^{ij}(\partial_i A_j - \partial_j A_i)\bar{\lambda}\bar{\lambda}\Big]\Big\}.\end{aligned}$$

In the exponent we recognize the first order lagrangian of the original theory after integrations over Π_1^α, $\bar{\Pi}_{1\dot{\alpha}}$, Π_2^α and $\bar{\Pi}_{2\dot{\alpha}}$ are performed in its path integral.

Let us adopt the normalization to write partition function of non–anticommutative $N=\frac{1}{2}$ supersymmetric $U(1)$ gauge theory as

$$\begin{aligned}Z_{NA} \;=\; &\int \mathcal{D}A_i\, \mathcal{D}P_i\, \mathcal{D}\lambda\, \mathcal{D}\bar{\lambda}\, \mathcal{D}D\, \mathcal{D}P\, \delta(D)\delta(P)\, \delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A}) \\ &\exp\Big\{ \frac{i}{\hbar} \int d^3x \Big[P^i \dot{A}_i + \dot{D}P - \frac{g^2}{2}(P_i)^2 - iC^{0i}P_i\bar{\lambda}\bar{\lambda} \\ &-\frac{1}{4g^2}(\partial_i A_j - \partial_j A_i)^2 - \frac{i}{g^2}\lambda\not{\partial}\bar{\lambda} + \frac{1}{2g^2}D^2 \\ &-\frac{i}{2g^2}C^{ij}(\partial_i A_j - \partial_j A_i)\bar{\lambda}\bar{\lambda}\Big]\Big\}.\end{aligned}$$

Therefore, by the applying the above transformation , partition function of its dual can be obtained as

$$
\begin{aligned}
Z_{NAD} = & \int \mathcal{D}A_i\, \mathcal{D}P_i\, \mathcal{D}\lambda\, \mathcal{D}\bar{\lambda}\, \mathcal{D}D\, \mathcal{D}P\, \delta(D)\delta(P)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A}) \\
& \exp\Big\{\frac{i}{\hbar}\int d^3x\Big[P^i\dot{A}_i + \dot{D}P - \frac{1}{2g^2}(P_i)^2 - \frac{ig^2}{2}C_D^{0i}P_i\bar{\lambda}\bar{\lambda} \\
& -\frac{4g^2}{4}(\partial_iA_j - \partial_jA_i)^2 - ig^2\lambda\not{\partial}\bar{\lambda} + \frac{g^2}{2}D^2 \\
& -\frac{ig^4}{2}C_D^{ij}(\partial_iA_j - \partial_jA_i)\bar{\lambda}\bar{\lambda}\Big]\Big\}.
\end{aligned}
$$

Here, we omitted the label "D" of the dual fields.

Comparing the results one concludes that the partition functions of non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory Z_{NA} and its dual Z_{NAD} are equivalent:

$$Z_{NA} = Z_{NAD}.$$

Therefore, under the strong-weak duality non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory is invariant.