Blackhole - string transition, AdS/CFT correspondence and critical unitary matrix models - I

Spenta R. Wadia
TIFR, India

P. Basu, L. Alvarez-Gaume, M. Marino, SRW

(to appear)

also: L. Alvarez-Gaume, C. Gomez, H. Liu + SRW

hep-th / 0502227

30 March 2006

12th Regional Conf. on Mathematical Physics

Islamabad

Blackholes are solutions of general relativity

10-dim. Schwarzschild solution

$$ds^2 = -dt^2\left(1-\frac{2GM}{r^7}\right) + dr^2\left(1-\frac{2GM}{r^7}\right)^{-1} + r^2 d\Omega_8^2$$

Horizon $r_h = (2GM)^{1/7}$, Curvature singularity at $r=0$

Euclidean continuation

$t \to i\tau \qquad -dt^2 = d\tau^2$

$$ds_E^2 = d\tau^2\left(1-\frac{2GM}{r^7}\right) + \underbrace{dr^2\left(1-\frac{2GM}{r^7}\right)^{-1}}_{d\rho^2} + r^2 d\Omega_8^2$$

$\downarrow$

$\rho \sim 0, \quad d(\beta\tau)^2 \rho^2$

$\Rightarrow \quad \beta \sim (GM)^{1/7} \sim \frac{1}{T}$

$$S = \frac{A_h}{4G} \sim \frac{(GM)^{8/7}}{G}$$

Bhs radiate : $M \to M - \Delta M$, $T \to T + \Delta T$, $r_h \to r_h - \Delta r_h$

Question: What is the description of the bh when $r_h \gtrsim l_s$, $T \gtrsim \frac{1}{l_s}$, $M \gtrsim \frac{1}{l_s}\cdot\frac{1}{g_s^2}$

Bh / String transition.

Strategy

10-dim. Sch. bh

$\downarrow$ in

$AdS_5 \times S^5 \longrightarrow$ Thermal $\mathcal{N}=4$, $SU(N)$ SYM on $S^3 \times S^1 = \partial(AdS_5)$

(S^1 ↑ temp.)

$\downarrow$

Order parameter (constant $SU(N)$ matrix)

$\downarrow$

Effective Lagrangian

$\downarrow$

large N phase transition + phases of gauge theory

$\downarrow$

Black hole $\longleftrightarrow$ gapped phase

Bh / string transition $\longleftrightarrow$ large N transition smoothened in double scaling limit. (crossover)

$\downarrow$

partition fn. at cross over

$$Z \sim i \exp(F(t)), \quad \ddot{F} = -f^2(t), \quad \ddot{f} = tf - f^3$$

small Blackhole state → state of strings

Susskind-Horowitz-Polchinski

10 dim. Sch. bh:

$$r_h^7 \simeq G_{10} M = g_s^2 l_s^8 M$$

for a fixed M, r_h decreases as g_s decreases.

$$r_h \sim l_s \Rightarrow g_s^2 \sim \frac{1}{M l_s}$$

GR description breaks down $R_{\mu\nu\rho\sigma} \sim O\left(\frac{1}{l_s^2}\right)$

$g_s^2 < \frac{1}{M l_s}$ horizon falls within the string length

- more appropriate to describe the <u>same state</u> in terms of fundamental strings (D-branes....

This transition is parametrically smooth and the entropy calculated in the 2-descriptions must match:

$$\underbrace{S_{bh}}_{r_h \sim l_s} \sim \underbrace{S_{str}}_{\text{string state at level } n} \Rightarrow \frac{r_h^8}{G} \sim \sqrt{n} \Rightarrow \frac{1}{g_s^2} \sim \sqrt{n}$$

$$\Rightarrow M l_s \sim \sqrt{n}$$

(string mass) at level n

10-dim. Sch. blackhole 'in' $AdS_5 \times S^5$

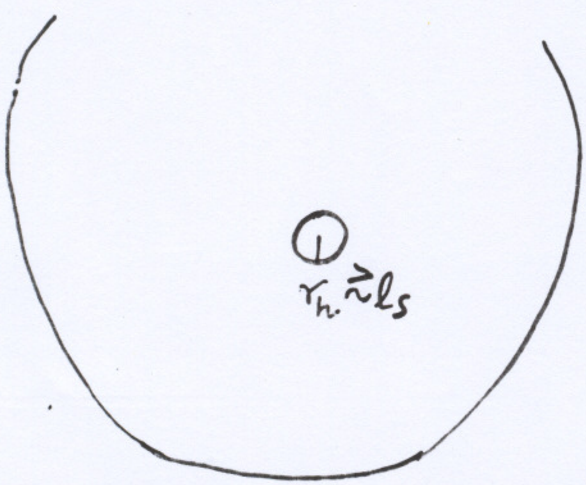
$AdS_5 \times S^5$:

$$ds^2 = -\left(\frac{\rho^2}{R^2}+1\right)dt^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{R^2}+1\right)} + \rho^2 d\Omega_3^2 + d\chi^2 + R^2 \sin^2\frac{\chi}{R}\, d\Omega_4^2$$

$$F^{(5)} = -\rho^3\, dt \wedge d\rho \wedge d\Omega_3 + R^4 \sin^4\frac{\chi}{R}\, d\chi \wedge d\Omega_4$$

$$\tilde{F}^{(5)} = \frac{N}{\pi^3 R^5} F^{(5)}, \quad \int_{S^5} \tilde{F}^{(5)} = N \quad \text{(units of 5-form flux)}$$

$\rho \ll R$ ds^2 is a flat 10-dim. metric.



$R \gg l_s$

$AdS_5 \times S^5$.

$F^{(5)}$

(Horowitz-Hubeny)

What about $F^{(5)}$ as one goes towards the bh?

$R_{\mu\nu\rho\sigma} \sim \frac{1}{l_s^2}$ is large near the horizon

Sufficient to treat $F^{(5)}$ in the backgnd

of the 10-dim. Sch. blackhole

Solutions:

$$\rho = r \sin\theta \qquad \rho^2 + x^2 = r^2$$
$$x = r \cos\theta$$

$\hookrightarrow$ 10-dim. radius

$$ds^2 = -\left(1 - \frac{r_h^7}{r^7}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r_h^7}{r^7}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\Omega_3^2 + \cos^2\theta\, d\Omega_4^2\right)$$

$$F^{(5)} = -\, g_1(r,\theta)\, r^3 \sin^4\theta\, dt \wedge dr \wedge d\Omega_3 - g_3(r,\theta)\, r^4 \sin^3\theta\, dt \wedge d\theta \wedge d\Omega$$
$$+\, g_2(r,\theta)\, r^4 \cos^5\theta\, dr \wedge d\Omega_4 - g_4(r,\theta) \sin\theta \cos^4\theta\, d\theta \wedge d\Omega_4$$

$g_i(r_h) = $ finite, only $g_1(r_h)$ needs to be specified

$r_h \sim l_s$

$$g_i(r \to \infty) = 1$$

No energy flux crosses the horizon:

$$T_{ab}\, k^a k^b = 0 \qquad k^a = \left(\frac{\partial}{\partial t}\right)^a$$

Black hole remains small

Some Thermodynamic formulas:

$$S_{bh} \sim \frac{A_h}{G} = \frac{r_h^8}{G} \sim \frac{(GM)^{8/7}}{G}$$

$$\frac{1}{T_{bh}} \sim \frac{\partial S}{\partial M} \sim (GM)^{1/7} \Rightarrow \frac{\partial T}{\partial M} < 0$$

$$r_+ \sim l_s \qquad (G = l_p^8 = g_s^2 l_s^8)$$

$$M_{bh} l_s \sim \frac{1}{g_s^2} \sim \frac{N^2}{\lambda^2}$$

$$\frac{l_p^8}{l_s^8} = g_s^2 \ll 1$$

$$S_{bh} \sim \frac{1}{g_s^2} \sim \frac{N^2}{\lambda^2}$$

$\lambda = g_s N$, 'tHooft coupling

$$T_{bh} \sim \frac{1}{l_s}$$

Conclusion:

Small Sch. blackhole is a state in λ type II B string theory and hence we can hope to discuss it in the dual $\mathcal{N}=4$ SYM theory on the boundary of $AdS_5 \times S^5$.

$$\text{boundary} \downarrow S^3 \times S^1$$

$\mathcal{N}=4$ SYM on $S^3 \times S^1$

$$S^3 \times S^1$$

radius: $\quad R \quad , \quad \frac{\tilde{\beta}}{2\pi}$

$$1 \quad , \quad \frac{\tilde{\beta}}{R} = \frac{1}{RT} \equiv \beta$$

by conformal symmetry

Since $T_{bh} \sim l_s^{-1}$, $\quad RT_b \sim R l_s^{-1} \gg 1$

$$Z = \sum_n e^{-E_n/RT}$$

spectrum of the gauge theory is discrete

(S^3 has unit radius)

and the energy gap $\sim O(1)$

Gauge group : $SU(N)$ $\quad (U(N))$

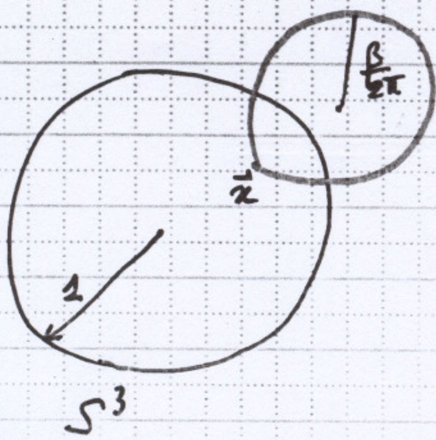
Coupling : $g_{YM}^2 = g_s$

$$\lambda = g_s N = \left(\frac{R}{l_s}\right)^4 \gg 1$$

g_s is small but finite

N is large but finite

Polyakov line as order parameter



$$U_{ab}(\vec{x}) = \left(T e^{i\int_0^\beta d\tau A_0(\vec{x},\tau)}\right)_{ab}$$

$N \times N$ unitary matrix

$\frac{tr}{N} U^n$ are gauge inv.

S^3 is compact + simply connected (radius ~ 1)

Integrate out all the fields

$A_i(\vec{x},\tau)$, $\phi^a(\vec{x},\tau)$, ~~$A_0(\vec{x},$~~ $\psi_\alpha(\vec{x},\tau)$ except

zero mode of $A_0(\vec{x},\tau)$.

Since S^1 is non-contractable and

$$\frac{tr}{N}\left[\exp\left(i\int_0^\beta A_0(\tau)\,d\tau\right)\right]^n \equiv \frac{tr}{N} U^n$$

are gauge invariant, $A_0(\tau)$ cannot be gauged away.

Effective action is a function of

$\frac{1}{N} tr U^n$, $\quad n = 0, \pm 1, \pm 2, \ldots\ldots, p$

The fact that the 10-dim blackhole sits at a point in S^5 may raise the question of SO(6) symmetry breaking and associated Nambu-Goldstone modes.

By analyzing the wave equation in the bh background one can show that the horizon bd. condition plus finite norm

$\Rightarrow$ a gap in the spectrum of the wave eqn.

The apparent SO(6) symmetry breaking is treated using standard collective coordinates.

Since zero modes are normalizable.

Effective Action

$$S_{eff}(U) = \sum_{i=1}^{p} a_i \, \mathrm{tr}U^i \, \mathrm{tr}U^{\dagger i}$$

$$+ \sum_{\vec{k},\vec{k}'} \alpha_{\vec{k},\vec{k}'} \Upsilon_{\vec{k}}(U) \Upsilon_{\vec{k}'}(U^\dagger)$$

$$\Upsilon_{\vec{k}}(U) = (\mathrm{tr}U)^{k_1} (\mathrm{tr}U^2)^{k_2} \cdots (\mathrm{tr}U^p)^{k_p}$$

$\vec{k} = (k_1, \ldots k_p)$

$\alpha_{\vec{k},\vec{k}'} = \alpha^*_{\vec{k}',\vec{k}}$ real

$\sum_j j k_j = \sum_j j k_j'$ each term $U(1)$ inv.

$\alpha_{\vec{k},\vec{k}'}$ + a_i are functions of λ, RT, N and possible vevs of the scalar fields ϕ^a.

e.g. of a typical term:

$$(\mathrm{tr}U)^2 \, \mathrm{tr}U^{\dagger 2} \, \mathrm{tr}U^2 \, \mathrm{tr}U^3 \, \mathrm{tr}U^{\dagger 6}$$

We can also introduce chemical potentials conjugate to the winding modes:

$$\sum_k b_k \, \mathrm{tr}U^k + \text{c.c.} = \int d\theta \, \Phi(\theta) \rho(\theta)$$

$a_i \; (\lambda, RT, N, \langle\Phi^2\rangle)$ and

$\alpha_{\vec{k},\vec{k}'} \; (\lambda, RT, N, \langle\Phi^2\rangle)$

contain detailed information about the gauge theory.

Difficult to calculate even in pert. theory (Ofer Aharony, Shiraz Minwalla et al)

We will assume that a_i, $\alpha_{\vec{k},\vec{k}'}$ are such that the partition function can be evaluated by a saddle pt. expansion in $\frac{1}{N}$ for N large but finite.

This will require a_i + $\alpha_{\vec{k}\vec{k}'}$ to have an expansion in $\frac{1}{N}$ and further in any particular term like

$$\mathrm{tr}\,U^{n_1} \ldots \mathrm{tr}\,U^{n_r}$$

$$\sum_{i=1}^{r} |n_i| \ll N^2 .$$

$$Z = \int dU \, e^{S_{eff}(U) + \sum_n b_n \operatorname{tr} U^n + c.c.}$$

$$\left\langle \frac{\operatorname{tr}}{N} U^n \right\rangle = \frac{\partial}{\partial b_n} \ln Z$$

As long as N is finite.

Z is a smooth function of all its parameters

However in order to make contact with the bulk theory (e.g. SUGRA) one must learn to extract 'finite N' physics in a saddle point expansion.

(more later)

Critical properties of the general unitary matrix model:

Lemma:

$$Z = \left(\frac{N^4}{2\pi^2}\right)^p \int_{-\infty}^{+\infty} \prod_{i=1}^{p} dg_i\, d\bar{g}_i\, d\mu_i\, d\bar{\mu}_i\; e^{N^2 \tilde{S}}\; \mathcal{Z}(g_i, \bar{g}_i)$$

$$\tilde{S} = -\sum_{j=1}^{p} a_j \mu_j \bar{\mu}_j + i \sum_j (\mu_j \bar{g}_j + \bar{\mu}_j g_j)$$

$$+ \sum_{\vec{k}, \vec{k}'} (-1)^{\sum_i k_i + \sum_j k'_j}\, Y_{\vec{k}}(\bar{\mu})\, Y_{\vec{k}'}(\mu)\, \alpha_{\vec{k}, \vec{k}'}$$

where $Y_{\vec{k}}(\bar{\mu}) = \prod_j \mu_j^{k_j} = \bar{\mu}_1^{k_1} \bar{\mu}_2^{k_2} \cdots \bar{\mu}_p^{k_p}$

$$\mathcal{Z} = \int dU\, e^{N \sum_{k=1}^{p} (g_k \, \mathrm{tr}\, U^k + \bar{g}_k \, \mathrm{tr}\, U^{\dagger k})}$$

The unitary matrix integral is now contained entirely in $\mathcal{Z}$.

This model can be analysed in great detail in the double scaling limit.

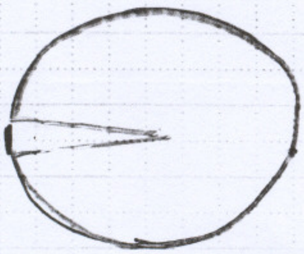
Critical phenomena of $\mathcal{Z}$ are reflected in Z.

Some results in the unitary matrix model

Density of eigenvalues:

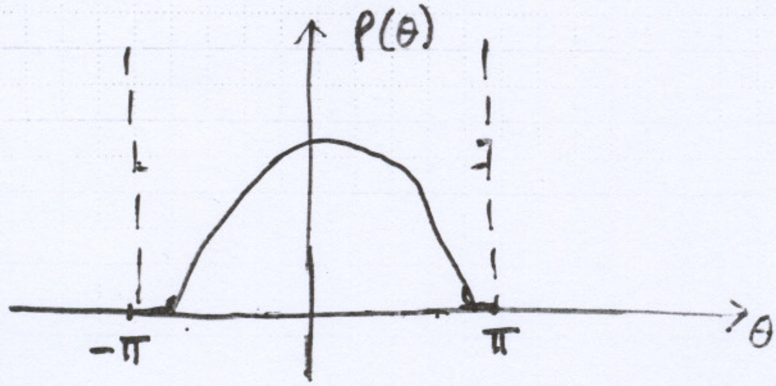
$$\rho(\theta) = \sum_n e^{in\theta} \frac{\mathrm{tr}}{N} U^n$$

At $N=\infty$ $\rho(\theta)$ can have a gap



$\rho(\theta)=0$ in the small arc

$\rho(\pi)=0$



generic behavior

⇓

e.g.
$$\rho(\theta) = \begin{cases} \frac{g}{\pi} \cos\frac{\theta}{2} \sqrt{g^{-1} - \sin^2\frac{\theta}{2}} & , g \geq 1 \quad \text{gapped} \\ \frac{1}{2\pi}\left(1 + g\cos\theta\right) & , g \leq 1 \quad \text{ungapped} \end{cases}$$

Important property of gapped phase

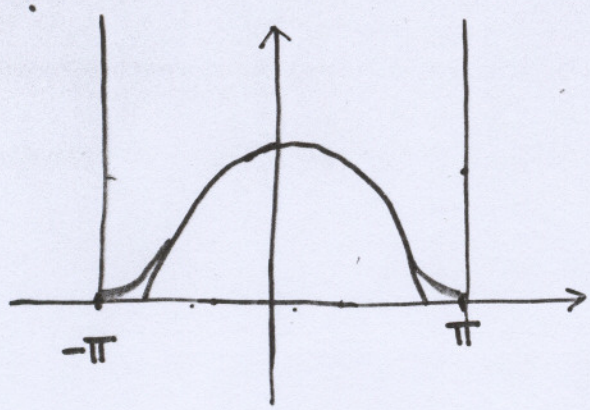
$$\left\langle \frac{\mathrm{tr}}{N} U^n \right\rangle \neq 0 \quad \forall\, n$$

As one varies the parameters of the theory,

the gap can become smaller and at $N = \infty$ one has a phase transition.

(Typically this is a 3rd order transition)

However it is possible to smoothen out this transition in a double scaling limit :



$\rho(\theta) \sim (\pi - \theta)^{2m}$ as $\theta \to 0$

each $m = 1, 2, \dots$ characterized a critical point in the space of couplings

e.g. for $m=1$, we can tune the temperature to reach it :

$$\left(\frac{T - T_c}{T_c} \right) = q N^{-\frac{2}{3}} \quad , \ q \text{ is fixed.}$$

as $T \to T_c$, N is tuned to hold q fixed.

$$N^2 F(g_k, \bar{g}_k) \equiv \ln Z$$

free energy of single trace matrix model.

$1/N$ expansion depends on phase:

Ungapped:

$$N^2 F = N^2 \sum_k \frac{g_k \bar{g}_k}{4} + e^{-2Nf(g_k,\bar{g}_k)}\left(\frac{1}{N} F_1^{(1)} + \frac{1}{N^2} F_2^{(1)} + \cdots\right.$$

gap opening as $\rho(\theta) \sim \theta^2$

$$N^2 F = N^2 \sum_k \frac{g_k \bar{g}_k}{4} + F_0^{(2)} + \frac{1}{N^{2/3}} F_1^{(2)} + \cdots$$

$$g - g_c \sim O(N^{-2/3})$$

gapped:

$$N^2 F = N^2 G(g_k, \bar{g}_k) + \sum_n \frac{G^{(n)}(g_k, \bar{g}_k)}{N^{2n}}$$

divergent at phase transition

this is how perturbation theory is organized in closed string theory

supergravity backgrounds correspond to the gapped phase

Modeling the bh→"string" transition.

- Behavior of large N perturbation theory ⇒ small bh sits in the gapped phase of the gauge theory.

- Solution in ungapped phase:

$$\rho(\theta) = \frac{1}{2\pi}\left(1 + \sum_l l g_l e^{il\theta} + c.c.\right)$$

gap opens at $\theta = \pi$, $\rho(\pi) = 0$

$$\Rightarrow \sum_k (-1)^k k\,(g_k^c + \bar{g}_k^c) = 1$$

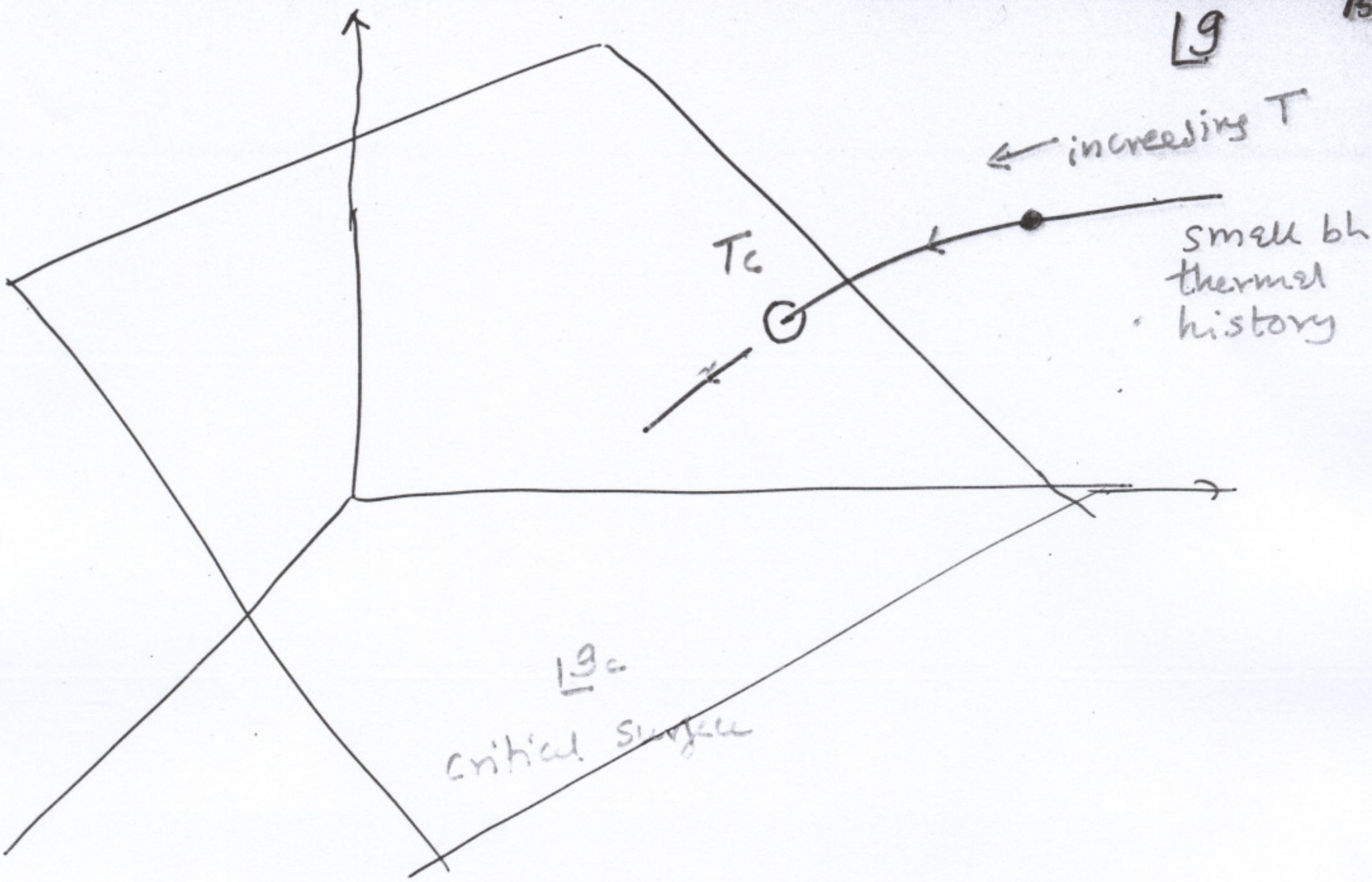
- At first critical point gap opens like

$$\rho_1(\theta) \sim (\pi-\theta)^2 \qquad \left(\rho_m(\theta) \sim (\pi-\theta)^{2m}\right)$$

- Operator that opens this gap:

$$\hat{O} = \sum_k \delta g_k \left(\mathrm{tr}\,U^k + \mathrm{tr}\,U^{-k}\right), \quad \delta g_k = N^{-2/3} t\, k\,(-1)^k$$

L9



As the temp. increases, the horizon shrinks, and the bh reaches $r_h \gtrsim l_s$.

In the gauge theory the couplings change as a function of T.

Assumption: at $T = T_c$ thermal history crosses the critical surface.

Then we can calculate the partition function in the nbd. of the critical surface and express it in terms of a renormalized coupling

Evaluation of Z near critical surface

The method involves evaluating $\mathcal{Z}$ and then one evaluates Z using a saddle point method in $\frac{1}{N}$.

$$Z \sim \int d\mu\, dg\, e^{N^2 \widetilde{S}(\mu, g) + N^2 F(g)}$$

$$F(g) = \sum_{\ell=1}^{p} \ell\, g_\ell \bar{g}_\ell + O\left(\frac{1}{N^2}\right)$$

$$\widetilde{S} = \underbrace{-\sum_i a_i |\mu_i|^2 + \sum_{\vec{k}, \vec{k}'} (-1)^{\sum_j k_j + \sum_i k_i'} Y_{\vec{k}}(\bar{\mu}) Y_{\vec{k}'}(\mu) \alpha_{k}}_{\equiv \mathcal{S}(\mu)}$$

$$+ i \sum_j (\bar{g}_j \mu_j + g_j \bar{\mu}_j)$$

The full action to extremize is:

$$\mathcal{S}(\mu) + i(g, \mu) + F(g)$$

$$\left(F(g) = \frac{1}{4}(g, g) + \frac{F_0}{N^2} + \frac{1}{N^{2+}} F \right.$$

$$\frac{\partial}{\partial \mu_j} \mathcal{S} = -i g_j \quad , \quad \frac{\partial}{\partial \bar{\mu}_j} \mathcal{S} = -i \bar{g}_j$$

$$g_j = -\frac{i}{j} \mu_j \quad , \quad \bar{g}_j = -\frac{i}{j} \bar{\mu}_j$$

Substituting $g_j = g_j^*$, $\bar{g}_j = \bar{g}_j^*$ we obtain a 'critical' hypersurface in the space of $(a_i, \alpha_{kk'})$

Perturbation expansion around saddle point near critical surface:

$$\mu \equiv \begin{pmatrix} \mu_j \\ \bar{\mu}_j \end{pmatrix}, \quad A = \begin{pmatrix} a_j \\ \alpha_{\vec{k},\vec{k}'} \end{pmatrix}, \quad g = \begin{pmatrix} g_j \\ \bar{g}_j \end{pmatrix}$$

$$\delta g = N^{-2/3} t$$

$$\delta \mu = N^{-4/3} n$$

$$\delta A = \tilde{g} N^{-2/3} \alpha, \quad \alpha = \left. \frac{\partial A}{\partial \beta} \right|_{\beta_c}$$

$$\tilde{g} = N^{2/3} (\beta_c - \beta)$$ ← renormalized coupling

$O(1)$ Z

$$Z \sim (\det{}^{-1/2}) e^{F(\vec{C} \cdot \vec{t})}$$

, $\vec{C}_k = k(-1)^k$

t_k depends on the parameters at $\beta = \beta_c$.

$F(t)$ is a smooth function of t, $(t \propto \tilde{g})$

$t << -1$

$$F(t) = \frac{t^3}{6} - \frac{1}{8}\ln(-t) - \frac{3}{128}\frac{1}{t^3} + \frac{63}{1024}\frac{1}{t^6} + \cdots$$

$t >> 1$

$$F(t) = \frac{1}{2\pi} e^{-\frac{4\sqrt{2}}{3} t^{3/2}} \left(-\frac{1}{8\sqrt{2}} \frac{1}{t^{3/2}} + \frac{35}{384}\frac{1}{t^3} + \cdots \right)$$

Condensation of Polyakov lines at crossover :

$$N^{-2/3} \left\langle \sum_k \frac{(-1)^k}{k} \left(\mathrm{tr}\, U^k + \mathrm{tr}\, U^{-k} \right) \right\rangle$$

$$\sim \frac{\partial}{\partial t} F(t)$$

Conclusions / Summary:

- The small Sch black hole can be considered a state in the gauge theory on the bd. of $AdS_5 \times S^5$. (S^5 is not crucial, can be X^5 with $U(1)$ isometry)

- Discrete spectrum & gap of the gauge theory enables a rep. of Z in terms of a constant thermal order parameter
$$U_{ab} \longrightarrow \left\{ \frac{\mathrm{tr}}{N} U^n \right\}$$

- We characterized a bh spacetime as the gapped phase of the unitary matrix model

- We proposed (on sound physical gnds.) the identification of the gap → gapless transition with bh → string transition

- Both transitions are smooth.
The transition in the gauge theory is smoothened using the double scaling limit.

- The specific heat is given by $\ddot{F} = -f^2$.
Condensation of Polyakov lines at crossover.

Open problem / Questions :

- What is the implication of this result for small Lorentzian bh?

- The question of resolution of curvature singularity of the bh. in the light of the fact that Z is finite.

- What is inside a blackhole ?