D-BRANES AND CATEGORY THEORY

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WHAT IS THE TALK ABOUT?

ACHARYA'S TALK YESTERDAY
KÖRNER'S TALK TODAY

MATHEMATICAL PHYSICS

THIS TALK TODAY

MATHEMATICAL

[AND NO, THE TITLE IS
<u>NOT</u> CHOSEN BECAUSE
TODAY IS THE 1st OF APRIL]

References:

Review by Aspinwall hep-th 0403166
Sharpe hep-th 0307245
Douglas ICTP Lectures
Spring School 2001

Papers by Douglas; Douglas, Fiol, Romelsberger;
[redacted] Suresh Govindarajan & T.J;
Suresh Govindarajan, T. Sarkar & T.J
Aspinwall & Lawrence
Aspinwall & Douglas.
⋮
and others.

Extension of derived category approach to branes on orientifolds ~
Suresh Govindarajan & T. Jayaraman
(in preparation).

GENERAL REMARKS:

SOLITON PHYSICS

$\Updownarrow$

MATHEMATICAL PHYSICS

Ex: VORTICES, MONOPOLES & INSTANTONS IN GAUGE THEORY.

SPECTACULAR APPLICATIONS IN MATHEMATICS

THIS TALK: ONE SUCH LINK BETWEEN DEEP MATHEMATICAL STRUCTURES AND SOLITONS OF STRING THEORY: D-BRANES.

D-BRANES: SUB-MANIFOLDS OF 10-DIM. SPACE-TIME ON WHICH OPEN STRINGS CAN END.

TODAY: CONSIDER D-BRANES WRAPPING NON-TRIVIAL CYCLES IN A COMPACTIFICATION $R^4 \times M$.

RESTRICT ATTENTION TO THE CASE WITH (SOME) UNBROKEN SUPERSYMMETRY (SPACE-TIME). IN THE SOLITON SECTOR.

SUCH SOLITONS ARE REFERRED TO AS BPS VIZ.

$M = |Z|$ where $Z \sim$ central charge in the SUSY algebra.

RESTRICT TO COMPLEX, KAHLER

$M \equiv$ CALABI-YAU MANIFOLD

RICCI-FLAT ; $C_1(M) = 0$.

(ALLOWED SUSY COMPACTIFICATION)

TWO KINDS OF BRANES

A & B :

$X^I(\sigma,\tau)$; $X^{\bar{I}}(\sigma,\tau)$ $\quad I=1,2,3$

$X^I = X^{\bar{I}} \Big|_{\sigma=0}$ for all $I=1,2,3$ — A

$X^I = 0$ for $I=1,2,3$. — B

σ-MODEL ACTION :

$$S = \int_\Sigma d^2z \Big\{ G_{I\bar{J}}\left(\partial X^I \bar{\partial} X^{\bar{J}} + \bar{\partial} X^I \partial X^{\bar{J}}\right) + iB_{I\bar{J}}\left(\partial X^I \bar{\partial} X^{\bar{J}} - \bar{\partial} X^I \partial X^{\bar{J}}\right)$$

$$+ iG_{I\bar{J}}\psi_-^{\bar{J}} D \psi_-^I + iG_{I\bar{J}}\psi_+^{\bar{J}} D \psi_+^I$$

$$+ R_{I\bar{I}J\bar{J}}\psi_+^I \psi_+^{\bar{I}} \psi_-^J \psi_-^{\bar{J}} \Big\}$$

TOPOLOGICAL TWISTING:

MODIFIES THE THEORY:

$\psi_\pm^{\bar J}$ ~ section of $X^*(\bar T_M)$

ψ_+^I ~ section of $K \otimes X^*(T_M)$

ψ_-^I ~ section of $\bar K \otimes X^*(T_M)$

Scalar: $\eta^{\bar J} = \psi_+^{\bar J} + \psi_-^{\bar J}$

$\theta_J = G_{J\bar K}\left(\psi_+^{\bar K} - \psi_-^{\bar K}\right)$

1-form: ρ^J ~ (1,0) part ψ_+^J

(0,1) part ψ_-^J

One SUSY op. $G \rightsquigarrow Q \; : \; Q^2 = 0$

Now $S = \int \{Q, V\} +$

$$\int \left(-\theta_J D\rho^J - \frac{1}{2} R_{J\bar J K\bar K}\, \rho^J \wedge \rho^K \; \theta_L G^{L\bar K} \right)$$

OBSERVABLES:

$$W_A = \eta^{\bar K_1}\cdots\eta^{\bar K_q}\, A^{J_1\cdots J_p}_{\bar K_1\cdots \bar K_q}\,\theta_{J_1}\cdots\theta_{J_p}$$

where

$$A = d\bar z^{\bar K_1}\cdots d\bar z^{\bar K_q}\, A^{J_1\cdots J_p}_{\bar K_1\cdots \bar K_q}\, \frac{\partial}{\partial z_{J_1}}\cdots\frac{\partial}{\partial z_{J_p}}$$

$A \sim (0,q)$ form valued in $\Lambda^p T_M$ $\quad \left(\begin{array}{l} Q\cdot X^I = 0 \\ Q X^I = \eta^I \end{array} \right)$

$$\{Q, W_A\} = -W_{\bar\partial A}$$

Q-coh. $\equiv$ Dolbeault cohomology on forms valued in $\Lambda^p T_M$

X's are constant maps.

When world-sheet has a boundary

$\theta_J = 0$ bdy. condns

(compatible with B-type bdy. condns)

GAUGE DEGREES OF FREEDOM:

In flat space ~ Chan – Paton Factors

In curved space ~ Vector bundles

Chan – Paton degrees of freedom

~ Maps from vector bundle to vect. bdl.

$$E_1 \longrightarrow E_2$$

Denoted $\underline{\mathrm{Hom}}(E_1, E_2)$

So finally open-string vertex ops in top. σ-model with boundary

$$H^{0,q}_{\bar\partial}\left(M, \underline{\mathrm{Hom}}(E_1, E_2)\right)$$

$$= \mathrm{Ext}^q(E_1, E_2) \leftarrow \text{sheaf cohomology.}$$

$q \equiv$ ghost number

SHEAVES & B-TYPE D-BRANES:

More general than vector bundles. Provides a unified way of ~~the~~ considering 6-branes, 4-branes, 2-branes, 0-branes all together.

A general D6-D4-D2-D0 configuration 9

≡ coherent sheaf

Nice vector bundles ≡ locally-free sheaves

(Sheaves considered here ~ sheaves of holomorphic functions)

All coherent sheaves can be built out of locally-free sheaves.

How do we do this?

1st: Vector bundles as D-branes

Charges ~ $Q(E) \sim ch(E)$

$$\sim rkE + c_1(E) + c_2(E) - \tfrac{1}{2}c_1^2(E) + \cdots$$

$ch_0(E) = rk(E) \equiv$ D6-charge

$ch_1(E) = c_1(E) \equiv$ D4-charge.

Ex: $E \equiv \mathcal{O}(+nD)$; $ch(E) = e^{nD}$

$$= 1 + nD + \tfrac{1}{2}n^2D^2 + \tfrac{1}{6}n^3D^3$$

D6-charge = +1

D4-charge = n

Now consider the complex

$$\mathcal{O}(-nD) \xrightarrow{f} \mathcal{O}$$

D6 = +1   D6 = +1

D4 = -1

$$f: \mathrm{Hom}(\mathcal{O}(-nD), \mathcal{O})$$
$$= \mathrm{Ext}^0(\mathcal{O}(-nD), \mathcal{O})$$

$q = 0$ ghost no. state

In the σ-model this corresponds to turning on a tachyon map

$$P \exp \int_{\partial\Sigma} [G, \phi^{\alpha\beta}] \qquad \phi^{\alpha\beta} = f$$

In the physical theory corresponding open string state has $\frac{1}{2} m^2 = \frac{1}{2}(q-1)$

So $q = 0 \Rightarrow$ tachyon.

Modified BRST charge

$$Q = Q_0 + f$$

Q-Cohomology expressed as follows:

$$0 \to \mathcal{O}(-nD) \xrightarrow{f} \mathcal{O} \longrightarrow \mathcal{O}_D \to 0$$

"Exact sequence"

For an exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \cdot \quad \ker g = \mathrm{Im} f$$

$$f \cdot g = 0$$

$$ch(A) - ch(B) + ch(C) = 0$$

Applying it to our example:

$\Rightarrow$ neighbouring elt. in the complex must be thought of as anti-brane.

Equiv. to imposing rel. $U(1)$ charge -1. Together with $+1$ charge of the operator

So,

$$ch\, \mathcal{O}_D = +D + \cdots \qquad rk = 0 \sim \text{D6-charge} = 0$$

D4-charge $= 1$.

So

<u>EXACT</u> <u>SEQUENCES</u> <u>DESCRIBE</u> <u>FORMATION</u> <u>OF</u> <u>BOUND</u> <u>STATES</u>

$$A + C \longrightarrow B \qquad B \equiv \mathrm{Ext}^1(A, C)$$

or $A + \bar{B} \to \bar{C}$

<u>Defn</u>: ~~Any~~ Every coherent sheaf has a resolun in terms of locally-free sheaves (vect. bdls.).

$\Rightarrow$

$$0 \longrightarrow \mathcal{F}_{-i} \xrightarrow{d_i} \mathcal{F}_{-i+1} \xrightarrow{d_{i-1}} \cdots \longrightarrow \mathcal{F}_0 \longrightarrow E \longrightarrow 0.$$

$d_{i-1} \circ d_i = 0$ / Exact everywhere.

So we can now see how to put all coherent sheaves (B-type D-branes) together.

Description is

coherent sheaves ~ Abelian category

open-string states ~ Morphisms between objects in the Abelian category.

<u>Category</u> ~ collection of objects $\{E\}$
<u>Morphisms</u> between the objects.

<u>Additive category</u> - Has a zero object $\equiv 0$.
0 such that $\mathrm{Hom}(0,B)$ & $\mathrm{Hom}(B,0)$ have precisely one morphism. If 0 exists, then for any B & C, the zero morphism in $\mathrm{Hom}(B,C)$ is the composition $B \to 0 \to C$

0-object for D-branes ~ vacuum (absence of a brane)

(0-morphism $\equiv$ id. map in $\mathrm{Hom}(B,C)$).

Abelian gp. structure on the set $\mathrm{Hom}(B,C)$ such that $(f+f')g = fg+f'g$; $f(g+g') = fg+fg'$

<u>Abelian category</u>: Every morphism has a kernel and a co-kernel.

$$A \xrightarrow{f} B \qquad \text{co-kernel} \equiv B/_{\mathrm{Im}\,f}$$

<u>Derived category</u>: (Of an abelian cat. $\mathcal{A}$)

1) Objects of $\mathcal{A}$
+ complexes of objects of $\mathcal{A}$.

2) $\mathcal{E}^{\bullet}$ (complex) $\quad \mathcal{E} \in \mathcal{A} \quad 0 \to \mathcal{E} \to 0$

Given $f : \mathcal{E}^{\bullet} \longrightarrow \mathcal{F}^{\bullet}$

$$f^n_{\bullet} : \mathcal{H}^n(\mathcal{E}^{\bullet}) \longrightarrow \mathcal{H}^n(\mathcal{F}^{\bullet})$$

If $f^n_{\bullet}$ is an isomorphism for all n, this is known as a quasi-isomorphism.

Identify all complexes that are quasi-isomorphic.

<u>Ex</u>: $\quad 0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow 0$

$$0 \longrightarrow \mathcal{O}_D \longrightarrow 0$$

3) Introduce $\mathcal{E}[n] \quad n \in \mathbb{Z}$
(R-charge copies)
$T^n(\mathcal{E}) \longrightarrow \mathcal{E}[n]$ (Shifts complexes by ~~one~~ n places to the left).

$\mathrm{Hom}(\mathcal{E}^\cdot[n], \mathcal{F}^\cdot[m])$
$= \mathrm{Ext}^{m-n}(\mathcal{E}^\cdot, \mathcal{F}^\cdot)$

Mess 2. No kernels & co-kernels in general. What is this good for?

Actually useful in describing what happens ~~xx~~ as we change Kahler moduli — phenomenon of bound states decaying, marginal stability, etc.

TRIANGLES & BOUND STATES.

$D^b(\mathrm{coh}\, M)$ ~ B-type branes, arbitrary deformations & open-string states.

Seq. of objects & morphisms $\triangle$

$\Delta : \quad A^\cdot \to B^\cdot \to C^\cdot \to A^\cdot[1]$

For any $A^\cdot, B^\cdot$, construct the mapping cone of $f : A^\cdot \to B^\cdot$ & a

$\Delta : \quad A^\cdot \xrightarrow{f} B^\cdot \to M(f) \to A[1]$

$M(f)^n = X^{n+1} \oplus Y^n$ & $d^n_{M(f)} = \begin{pmatrix} -d_A^{n+1} & \\ f^{n+1} & d_Y^n \end{pmatrix}$

Std. Δ: $X \xrightarrow{f} Y \to M(f) \to X[1]$

Distinguished Δ: Δ isomorphic to std. Δ

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr, "a"'] & & B \arrow[ul, "b"'] \end{tikzcd}$$

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$$

also

$$\begin{tikzcd} & C \arrow[dl, "c"'] & \\ A[1] \arrow[rr, "-a[1]", "{[1]}"'] & & B \arrow[ul, "b"'] \end{tikzcd}$$

$$B \xrightarrow{b} C \xrightarrow{c} A[1] \xrightarrow{-a[1]} B[1]$$

Compare to $\quad 0 \to A \to B \to C \to 0$

$B \equiv \mathrm{Ext}^1(C, A)$

Same as $\mathrm{Hom}(C, A[1])$ in D^b.

Interpret this as A & C bind through a potentially tachyonic open string to give B.

Central charge $Z_{\mathcal{E}^\cdot} = \int \exp -(B+iJ)\, ch(\mathcal{E}^\cdot) \sqrt{td\, M}$

Generally $Z = \int ch(\mathcal{E}^\cdot) \wedge \Pi$

Choose a grading for $\mathcal{E}^{\cdot}$ s.t.

$$\xi(\mathcal{E}^{\cdot}) = \frac{1}{\pi} \arg Z(\mathcal{E}^{\cdot})$$

$$\xi(\mathcal{E}^{\cdot}[n]) = \xi(\mathcal{E}^{\cdot}) + n$$

In the Δ, if A & B are stable then C is stable ~~only~~ iff $\xi(B) < \xi(A) + 1$

Marginally stable $\xi(B) = \xi(A) + 1$
& $\xi(C) = \xi(B) = \xi(A) + 1$.

If A & C are stable then B is stable if $\xi(A) < \xi(C)$

Translation to the derived category of the original Hom condition in the exact sequence

[In general note $\xi(A) > \xi(B) \Rightarrow$
Hom $(A, B) = 0$
Otherwise states with $h < 0$ ($\because q < 0$)]

If flow of grading changes this condition then bound state becomes unstable.

FLOPS & 0-BRANES :

Consider a non-compact CY
$\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbb{CP}^1$

Flop ~ $CP^1 \longrightarrow$ goes to zero size
and then a new CP^1 emerges
out of the 2 lines $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Good example since Π & hence
Z have exact expression over all of
Kahler moduli space which is simple

Consider $\mathcal{O}_C(m)$ (where $C \equiv \mathbb{CP}^1$)

$$Z(\mathcal{O}_C^{(m)}) = -t + m + 1 \qquad t = \int_C B + iJ$$

Consider a 0-brane on $\mathbb{CP}^1$

$$0 \longrightarrow \mathcal{O}_C(-1) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_x \longrightarrow 0 .$$

Δ :

$$\begin{tikzcd}
 & \mathcal{O}_C(-1)[1] \arrow[dl, "\theta/\pi"', "{[1]}"] & \\
\mathcal{O}_C \arrow[rr, "1-\theta/\pi"'] & & \mathcal{O}_x \arrow[ul, "0"']
\end{tikzcd}$$

In the neighbd. of $t \to 1$ (where $\mathbb{CP}^1 \to 0$ size
$t = 1 + \epsilon e^{i\theta}$

As $\theta > \pi$

$\mathcal{O}_x$ decays into $\mathcal{O}_C$ & $\mathcal{O}_C(-1)[1]$

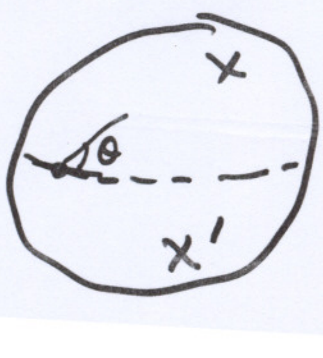
(Stable only when the opp. side is grade < 1)

Also

$$\begin{tikzcd} & \mathcal{O}_C[1] \arrow[dl, dash, "1-\theta/\pi"'] \arrow[dr, dash, "0"] & \\ \mathcal{O}_C(1) \arrow[rr, dash, "\theta/\pi"'] & & \mathcal{O}_x \end{tikzcd}$$

For $\theta < 0$, $\mathcal{O}_x$ decays to $\mathcal{O}_C(1)$ & $\mathcal{O}_C[1]$.

Now how does the stable 0-brane of the new 'emerging' $\mathbb{CP}^1$ come about?



Using $\operatorname{Ext}^n(\mathcal{O}_C, \mathcal{O}_C(-1))$

$$= \mathbb{C}^2 \quad \text{if } n = 2,3$$

$$= 0 \quad \text{if } n \neq 2,3$$

we may use $f \in \operatorname{Hom}(\mathcal{O}_C[-1], \mathcal{O}_C(-1)[1])$
$\simeq \mathbb{C}^2$

to form

$$\begin{tikzcd}[column sep=small]
 & D_f \arrow[dr, dash, "\theta"] & \\
\mathcal{O}_C[-1] \arrow[ur, dash, "\theta/\pi - 1", "{[1]}"'] \arrow[rr, dash, "f", "2-\theta/\pi"'] & & \mathcal{O}_C(-1)[1]
\end{tikzcd}$$

D_f is stable as $\theta > \pi$

can re-scale f by a complex no.

So a $\mathbb{P}^1$ worth of D_f. These are

the pts. D_f of the new $C\mathbb{P}^1$.

OTHER USES OF THE DERIVED CATEGORY PICTURE:

i) Relation between diff. natural descriptions of B-branes at diff. points in Kahler moduli space.

$$C^3/\mathbb{Z}_3 \longrightarrow \mathcal{O}(-3) \text{ on } \mathbb{P}^2$$

Quiver representation $\cong D^b(\text{R-mod}) \simeq D^b(\text{coh } X)$

Presented as a derived equivalence

But different Abelian categories in the two. Hence different physical branes in the 2 regions

ii) Understanding Seiberg dualities as derived equivalences again.

iii) Computing monodromy.

TOWARDS A CONCLUSION

~ Interesting influence on questions in mathematics.

(Bridgeland, ...)