

BPS M-Brane Geometries

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12th Regional Conference on Mathematical Physics,
National Centre for Physics, Islamabad.

28 March 2006

Work done in collaboration with Ansar Fayyazuddin

String Theory hasn't been around that long, but it is already considered 'text-book material' that special holonomy manifolds are, well, special.

1 Introduction

Special holonomy plays a prominent role in string theory and M-theory primarily because the simplest vacua preserving some fraction of supersymmetry are compactifications on manifolds of special holonomy. The case that has received the most intensive study is Calabi-Yau three-folds (CY_3), first because heterotic string compactifications on such manifolds provided the first semi-realistic models of particle phenomenology, and second because type II strings on Calabi-Yau three-folds exhibit the seemingly miraculous property of "mirror symmetry." Recently, seven-manifolds with G_2 holonomy have received considerable attention, both because they provide the simplest way to compactify M-theory to four dimensions with $\mathcal{N} = 1$ supersymmetry, and because of some unexpected connections with strongly coupled gauge theory.

Just so we can understand why special holonomy manifolds are distinguished from the crowd, let's try and trace the roots of this all important property they have, of preserving supersymmetry.

And since we have the option of going up to 11 dimensions, why stay in 10?

So for the rest of this talk, we're going to talk of M-Theory, or 11-dimensional supergravity, to be more honest!

D =11 Supergravity

The bosonic action of 11-dimensional supergravity is given by

$$S = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-G} R - \frac{1}{2} F \wedge *F - \frac{1}{6} A \wedge F \wedge F$$

and the equations of motion are

$$R_{IJ} = \frac{1}{12} F_{IKLM} F_J{}^{KLM} - \frac{1}{144} G_{IJ} F^{KLMN} F_{KLMN}$$

$$d * F + \frac{1}{2} F \wedge F = 0$$

$$dF = 0$$

The bosonic fields are the metric and a three-form gauge potential A (whose associated field strength we denote by F)

There is, in addition, a single fermionic field -- the gravitino Ψ .

In purely geometric backgrounds, with no field strength flux,

$$R_{IJ} = \frac{1}{12} F_{IKLM} F_J{}^{KLM} - \frac{1}{144} G_{IJ} F^{KLMN} F_{KLMN}$$

$$\delta_\chi \Psi_I = (\partial_I + \frac{1}{4} \omega_I^{ij} \hat{\Gamma}_{ij} + \frac{1}{144} \Gamma_I{}^{JKLM} F_{JKLM} - \frac{1}{18} \Gamma^{JKL} F_{IJKL}) \chi = 0.$$

$$F = 0 \Rightarrow R_{IJ} = 0 \text{ and } \nabla_I \chi = [\partial_I + \frac{1}{4} \omega_I^{ij} \hat{\Gamma}_{ij}] \chi = 0$$

$$\text{This implies } [\nabla_I, \nabla_J] \chi = \frac{1}{4} R_{IJKL} \Gamma^{KL} \chi = 0$$

So the spinor has to be a singlet under the Spin(1,10) subgroup H generated by $R_{IJKL} \Gamma^{KL}$. This constrains possible choices of H - the reduced holonomy group.

$$H = \text{Spin}(7) \text{ when } d = 8,$$

$$H = G_2 \text{ when } d = 7$$

$$H = \text{SU}(n) \text{ when } d=2n,$$

$$H = \text{Sp}(n) \text{ when } d=4n$$

Neat as this classification scheme is, most of the situations we have to deal with involve a non-zero field strength flux!

M-Branes are charged, gravitating objects which modify the background into which they are placed.

Calabi-Yau spaces have supersymmetric cycles. But once an M-brane is wrapped on such a cycle, the back-reaction of the brane changes the geometry such that the manifold no longer remains Calabi-Yau.

$$R_{IJ} = \frac{1}{12} F_{IKLM} F_J{}^{KLM} - \frac{1}{144} G_{IJ} F^{KLMN} F_{KLMN}$$

➡ Not Ricci-flat anymore!

$$\delta_\chi \Psi_I = (\partial_I + \frac{1}{4} \omega_I^{ij} \hat{\Gamma}_{ij} + \frac{1}{144} \Gamma_I{}^{JKLM} F_{JKLM} - \frac{1}{18} \Gamma^{JKL} F_{IJKL}) \chi = 0.$$



Spinors are no longer covariantly constant !

We want to study and hopefully to classify these new, supersymmetric 'back-reacted' geometries.

Requiring Supersymmetry Preservation

Our goal is to look for bosonic supersymmetric solutions to 11-dimensional supergravity.

Since we require the fermions to vanish, it is clear that

$$\delta_{\text{susy}}(\text{bosons}) = \text{fermions} = 0$$

Hence, to ensure a supersymmetric solution, we need only to impose that

$$\delta_{\text{susy}} \Psi = 0$$

$$\delta_{\chi} \Psi_I = \left(\partial_I + \frac{1}{4} \omega_I^{ij} \hat{\Gamma}_{ij} + \frac{1}{144} \Gamma_I^{JKLM} F_{JKLM} - \frac{1}{18} \Gamma^{JKL} F_{IJKL} \right) \chi = 0.$$

It turns out that if a metric and field strength satisfy this equation, they also provide a solution to Einstein's equations.

(as long as the Bianchi Identity and equations of motion for F are satisfied)

Supergravity Solution for a Planar M5-brane

A flat M5-brane with worldvolume 012345 is a half-BPS object.

It preserves 16 supersymmetries corresponding to the components of the spinor χ which obey the projection

$$\hat{\Gamma}_{\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5}\chi = \chi.$$

Since the M5-brane is a charged massive object, it deforms the flat space in which it is placed such that the metric is given by

$$ds^2 = H^{-1/3}\eta_{\mu\nu}dX^\mu dX^\nu + H^{2/3}\delta_{\alpha\beta}dX^\alpha dX^\beta$$

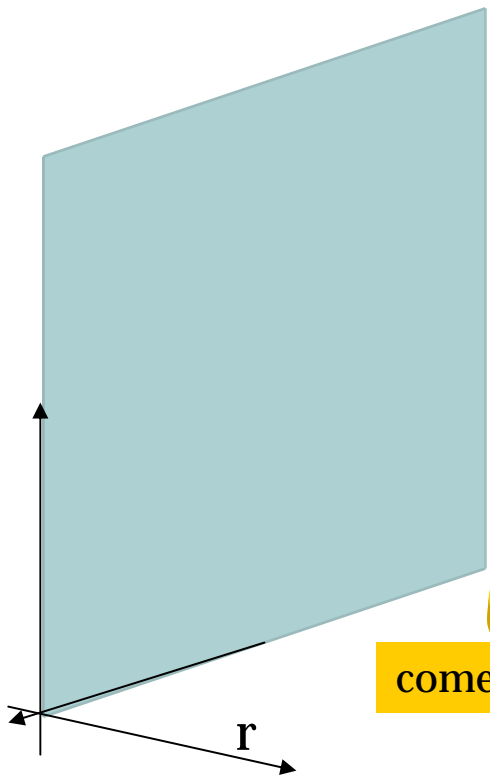
$$H = 1 + \frac{a}{r^3}$$

radial coordinate in transverse directions X^α

comes from imposing $d^*F = 0$

and the field strength by

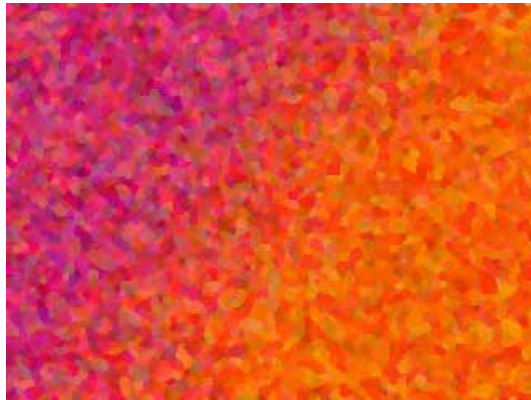
$$F_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta\rho}\partial_\rho H.$$



The Geometry of an M-Brane Wrapping a Supersymmetric Cycle

Common directions

X^μ



- Brane spans entire space
- Expect Poincare Invariance

Relative transverse directions

Z^m



- Define complex coordinates
- Brane wraps supersymmetric cycle in this space

Overall transverse directions

X^α



- Brane appears point-like
- Expect rotational invariance

M5-branes wrapping holomorphic cycles

Holomorphic cycles are known to be supersymmetric. They are in fact, perhaps the simplest supersymmetric cycles, so we consider them first.

Based on the isometries expected of the brane configuration, Fayyazuddin and Smith came up with an ansatz for the metric.

$$ds^2 = H_1^2 \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{M\bar{N}} dz^M dz^{\bar{N}} + H_2^2 \delta_{\alpha\beta} dX^\alpha dX^\beta$$

■ In order to preserve translational invariance along the common directions, H_1 , H_2 and the metric $G_{M\bar{N}}$ must be independent of X^m

■ We find that supersymmetry preservation requires

$$H \equiv H_1^{-3} = H_2^6$$

The Plan of Attack

- Find Killing Spinors of the background by solving

$$\chi = \frac{1}{p!} \frac{1}{\sqrt{h}} \hat{\epsilon}^{\alpha_1 \dots \alpha_p} \Gamma_{M_1 \dots M_p} \partial_{\alpha_1} X^{M_1} \dots \partial_{\alpha_p} X^{M_p} \chi.$$

Write these solutions in terms of Fock space states.

- Impose supersymmetry preservation to obtain a set of constraints.

Solve these to express the field strength and functions in the ansatz, in terms of the metric in the ‘relative transverse’ or embedding space.

This metric is ‘known’ modulo solution of a non-linear differential equation.

However, even in the absence of an explicit expression for the metric, we are able to find a constraint which it must obey.

This constraint serves to characterize the manifold – in particular, its departure from Special Holonomy.

M5-brane wrapping a holomorphic 2-cycle in C^2

Metric:
$$ds^2 = \underbrace{H_1^2 \eta_{\mu\nu} dX^\mu dX^\nu}_{0\ 1\ 2\ 3} + 2 \underbrace{G_{M\bar{N}} dz^M dz^{\bar{N}}}_{u, v} + \underbrace{H_2^2 \delta_{\alpha\beta} dX^\alpha dX^\beta}_{8, 9, 10}$$

$det G_{M\bar{N}} = \sqrt{G} = H^{1/3}$

Killing spinor $\chi = \alpha + C\alpha^*$ $\alpha = H^{-1/12} \eta |00\rangle$ where $i\Gamma_{0123}\eta = \eta$

The four-form field strength can be determined from A.

$$A = \frac{1}{2} H^{-2/3} dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge \omega$$

$$d^*F = 0$$

$dF=0$ implies
$$\partial_\alpha^2 g_{M\bar{N}} + 2 \partial_M \partial_{\bar{N}} H = 0 \quad g_{M\bar{N}} = H^{1/3} G_{M\bar{N}}$$

Supersymmetry preservation imposes a constraint on the complex submanifold

$$d[H^{1/3} \omega] = 0$$

U5 wrapping S_2 in D^3

$$ds^2 = H^{-1/3} (-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + 2G_{M\bar{N}} dz^M dz^{\bar{N}} + H^{2/3} dx_{10}^2$$

0	1	2	3	4	5	6	7	8	9	10	
x	x	x	x	not a cycle							

- $\det G_{M\bar{N}} = \sqrt{G} = H$
- $A_{0123N\bar{N}} = i H^{-2/3} G_{M\bar{N}}$

$$A = H^{-2/3} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega_G$$

Metric constraint:

$$\partial [H^{-1/3} \omega_G \wedge \omega_G] = 0 \quad \text{IMPORTANT!!}$$

analogous constraints hold for M2 branes also.
 $[\partial \pm \varphi] = 0$

The manifold is evidently not Kähler — cannot be Calabi-Yau!

$G_{M\bar{N}}$ can, in principle be found by solving a differential equation.

U5 Wrapping S_4 in D^3

0	1	2	3	4	5	6	7	8	9	10	
x	x	not a cycle									

$$ds^2 = H^{-1/3} (-dx_0^2 + dx_1^2) + 2G_{M\bar{N}} dz^M dz^{\bar{N}} + H^{2/3} (dx_8^2 + dx_9^2 + dx_{10}^2)$$

- $\det G = \text{constant}$
- $A = H^{-1/3} \omega \wedge \omega \wedge dx_0 \wedge dx_1$
- metric constraint: $\partial [H^{1/3} G_{M\bar{N}}] = 0$

warped Kähler condition

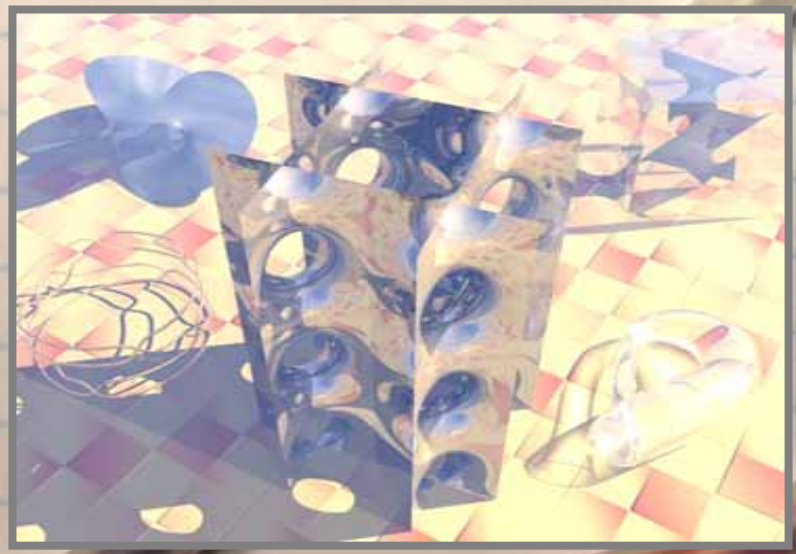
$G_{M\bar{N}}$ subject to non-linear differential equation

faces.

$\mathbb{Z} + \mathbb{Z} \cdot B$
①

$$ds_{11}^2 = e^{2\alpha} ds_4^2$$
$$ds_4^2 = H^{-1/3}$$

- Can continue for M2 branes wrapping holomorphic cycles,
- M5 branes wrapping special lagrangian cycles...



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Special Lagrangian Cycles

If L is a real n -dimensional sub-manifold of C^n and $\omega|_L = 0$,
 L is said to be Lagrangian.



Symplectic form;
Kahler form on
Calabi-Yau

If, in addition, $Im \Omega|_L = 0$, L is known as a Special Lagrangian cycle.

For such a cycle, $\Omega|_L = Re \Omega|_L = Vol(L)$

And we say that L is calibrated by $Re \Omega$

More generally, we would say that L was calibrated by $\Omega e^{i\theta}$ if

$$[Cos\Theta Re \Omega + Sin\Theta Im \Omega]|_L = Vol(L)$$

Let's make this more concrete by considering a simple example

The defining equations for a Special Lagrangian cycle are

$$\omega|_{\mathcal{L}} = 0 \quad \text{Im } \Omega|_{\mathcal{L}} = 0 \quad \text{Re } \Omega|_{\mathcal{L}} = \text{Vol}(\mathcal{L})$$



We find that

$$\tilde{\omega} = \text{Re}\Omega \quad \text{Im}\tilde{\Omega} = -\text{Im}\Omega \quad \text{Re}\tilde{\Omega} = \omega$$

$$\omega|_{\mathcal{L}} = 0 \quad \text{Im } \Omega|_{\mathcal{L}} = 0 \quad \text{Re } \Omega|_{\mathcal{L}} = \text{Vol}(\mathcal{L})$$



$$\tilde{\omega} = \text{Vol}(\mathcal{L}) \quad \tilde{\Omega}|_{\mathcal{L}} = du \wedge dv|_{\mathcal{L}} = 0$$

In order to be Special Lagrangian in z, w a cycle must be holomorphic in u, v

M5-brane wrapping a Special Lagrangian 3-cycle

Based on the isometries expected of the brane configuration, we can write down an ansatz for the metric.

$$ds^2 = H_1^2 \eta_{\mu\nu} dx^\mu dx^\nu + G_{IJ} dy^I dy^J + H_2^2 \delta_{\alpha\beta} dx^\alpha dx^\beta$$

- Once again, translational invariance implies that all functions in the ansatz are independent of X^μ
- We find that supersymmetry preservation requires

$$H \equiv H_1^{-3} = H_2^6$$

- The Killing Spinor is

$$\epsilon = H^{-1/12} \hat{\psi} \otimes \eta_{000} - \gamma_5 H^{-1/12} i\gamma_3 \gamma_5 \hat{\psi} \otimes \eta_{111}$$

where $\psi^* = i\gamma_3 \psi$ and $\hat{\psi}$ is a constant spinor.

This expression allows us to construct both F and $*F$

The gauge potential which couples to the M5-brane is given by

$$A = \frac{1}{2} H^{-1/2} dX^0 \wedge dX^1 \wedge dX^2 \wedge \text{Re } \Omega$$

provided $H = G$ and $d_6(\Omega - \bar{\Omega}) = 0$ where Ω is a (3,0) form.

$\text{Im } \Omega$, a form which is orthogonal to the calibration $\text{Re } \Omega$, is closed.

For backgrounds in which M-branes wrapped holomorphic cycles, the geometry was also modified in such a way that a form orthogonal to the calibration was closed.

Due to the presence of an M5-brane wrapping the Special Lagrangian 3-fold, **only an almost complex structure survives** in what was once the Calabi-Yau, but this is sufficient to define the notion of (p,q) tensors.

We can define a (1,1) form,

$$B_{IJ} = \eta_{m\bar{n}}(e_I^m e_J^{\bar{n}} - e_I^{\bar{n}} e_J^m).$$

by lowering an index on the almost complex structure.

B is the analogue of the Kahler form on our almost complex manifold.

B is not closed, and in fact is required by supersymmetry to obey the following constraints

$$\partial[H^{1/6}B \wedge B] = 0$$

$$\partial_I[H^{1/2}B^I{}_J] = 0$$

In addition to the above, supersymmetry also requires that

$$\Omega \wedge dB = 0$$



d could carry index α or I

$$\bar{\Omega}^{IJK} \partial_\alpha \Omega_{IJK} = 12 \partial_\alpha \ln H$$

$$\bar{\Omega} \wedge *_6 d_6 \Omega = 0$$



$d\Omega$ has no (3,1) part!

Conclusions

We study the supergravity backgrounds of M-branes wrapping supersymmetric cycles.

Although we do not write down explicit metrics, we are able to classify the resulting ‘back-reacted’ geometries through a set of constraints on the characteristic differential forms of the manifolds in question.

An example of one such constraint is that we find the Hodge dual of the calibrating form is always closed.

$$A = \frac{1}{2} H^{-2/3} dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge \omega$$

$$d[H^{1/3} \omega] = 0$$

$$A = \frac{1}{2} H^{-2/3} dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge \omega$$

$$d[H^{-1/3} \omega \wedge \omega]$$

$$A = \frac{1}{2} H^{-1/3} dX^0 \wedge dX^1 \wedge \omega \wedge \omega$$

$$d[H^{2/3} \omega] = 0$$

$$A = \frac{1}{2} H^{-1/2} dX^0 \wedge dX^1 \wedge dX^2 \wedge \text{Re } \Omega$$

$$d_6(\Omega - \bar{\Omega}) = 0$$