Quantum ElectroDynamics As I Look into it

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QED

Ingredients

- Electrons, Positrons
- Photons

Recipe

Dirac equation to describe electron and positron.

Maxwell equation for photon.

Transformation of the field representing a particle or system (symmetry concept/gauge invariance)

Result

Feynman diagrams (Tool/Device)

Out come of result

Quantum mechanical amplitudes based on perturbation theory to calculate cross-section

The rate for scattering processes

This is the Plan/Outline of my Lectures

The Electrons

What do you know about electron?

Coloumbs Law

Yes!

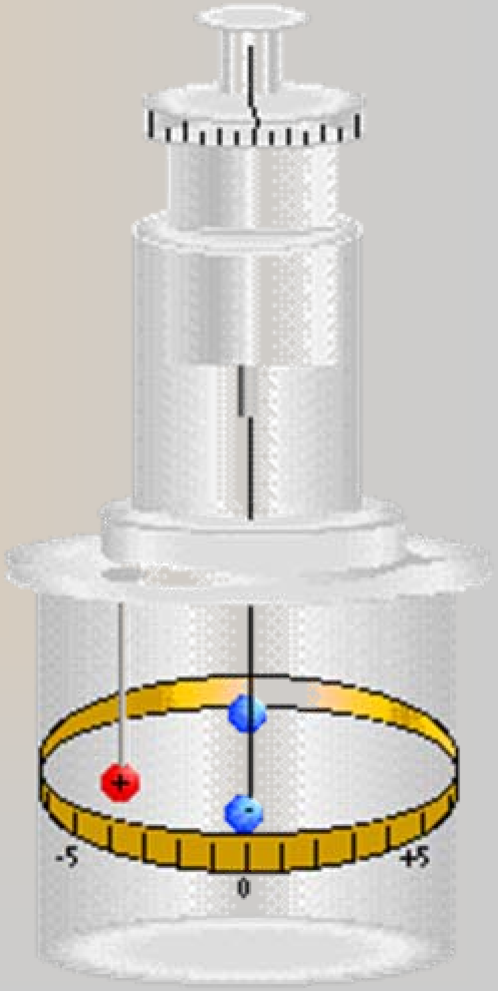
Coloumbs Law

$$F = \frac{kq_1 q_2}{r^2}$$

- *The Coloumbs interaction of two electrons at separation r is*

$$E_{Coulomb} = \frac{e^2}{4\pi r}$$

Magnitude of electron charge



What's New in Old

Simple idea great information

- *In High Energy Physics (HEP),* $\hbar = c = 1$

$$M = [E][c]^{-2} \qquad L = [\hbar][c][E]^{-1} \qquad T = [\hbar][E]^{-1}$$

- *Length and reciprocal Energy has the same dimension*

$$Er = \frac{e^2}{4\pi} \equiv \alpha = \frac{1}{137.03604} = 0.0073$$

Strength of the interaction between two electrons

dimensionless

Sommerfeld fine structure constant

0.000053

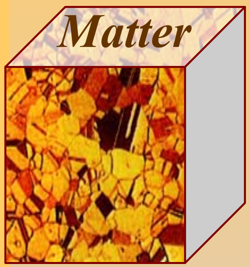
Let's start our formal journey

Novel Idea

There must be some set of smallest constituent parts, which are the building blocks of all matter.

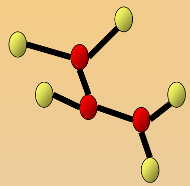
Weinberg Reduction Approach

Matter



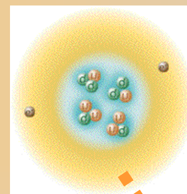
10^{-2}m

Molecule



10^{-9}m

Atom

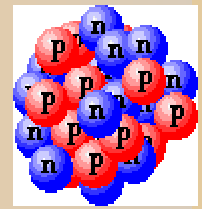


10^{-10}m

Condensed matter/Nano-Science/Chemistry

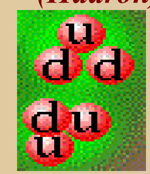
Atomic Physics

Nucleus



10^{-14}m

Baryon
(Hadron)



protons, neutrons,
mesons, etc.
p,Σ,π...

Quark

u

6

top, bottom,
charm, strange,
up, down

Electron
(Lepton)

6

~10^{-18}m

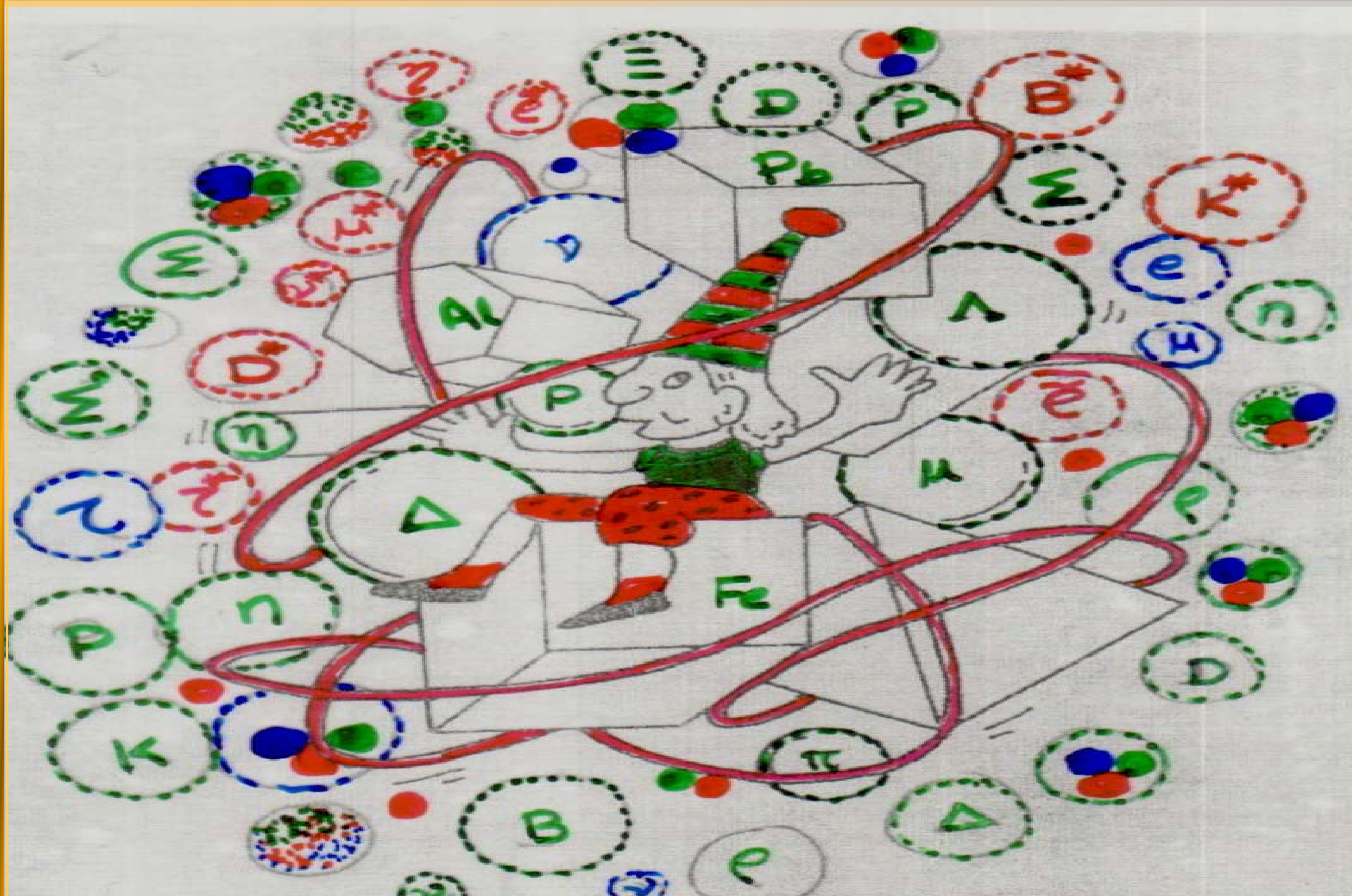
Electron, Muon , tau
ν_e , ν_μ , ν_τ

High Energy Physics

Thanks to ***experimentalists***

we are able to say some thing about

"Set of smallest constituents of matter"


Pb
Al
Fe

Building Blocks Of Matter

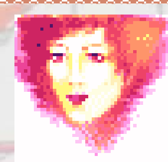
Lepton

move freely

Quark

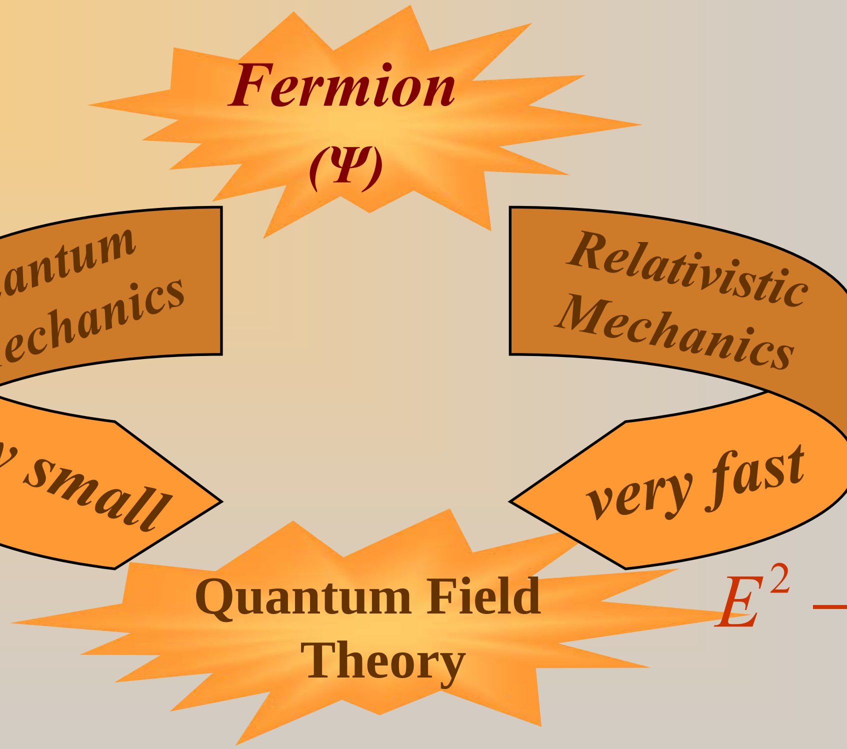
confined

Fermions(1/2) → *(point)*



Represented by Ψ

Frame work



$$p = \frac{h}{\lambda}$$

$$E^2 - p^2c^2 = m^2c^4$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$P^\mu = (E, p)$$

$$p \rightarrow -i\hbar \nabla = -i\hbar \frac{\partial}{\partial x}$$

$$P^\mu = i\hbar \partial^\mu$$

$$\frac{\partial}{\partial x_\mu}$$

Klein-Gordon Equation

- Relativistic Energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4$$

$i\hbar\partial^{\mu}$

- In four vector notation

$$P^{\mu}P_{\mu} - m^2c^4 = 0$$

$$\frac{\partial}{\partial x_{\mu}}$$

$$\left(-\hbar^2\underbrace{\partial^{\mu}\partial_{\mu}}_{\Box} - m^2c^2\right)\Psi = 0$$

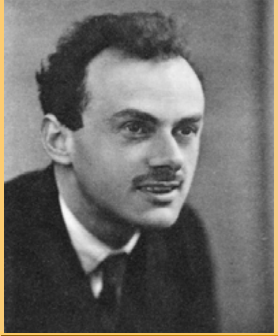
- In time space component form

$$-\frac{1}{c^2}\left(\frac{\partial^2\Psi}{\partial t^2}\right) + \nabla^2\Psi = \left(\frac{mc}{\hbar}\right)^2\Psi$$

2nd order in time

Comment on Klein-Gordon Eq

- ***Schrodinger discoverd*** *this eq.* ***before non-relativistic*** *(which is on his name)*
- *But,* ***rejected*** *due to non-compatibility with* ***statistical interpretation*** *of Ψ.*
- ***Pauli and Weisskopf*** *(1934) showed that the* ***statistical interpretation*** *itself has* ***flaw*** *in relativistic quantum theory.*
- ***Restored*** *the* ***Klein-Gordon*** *equation to it rightful place.*
- *Keep* ***Dirac*** *equation for* ***spin ½ particles***



Dirac's strategy

- *To start with KG eq. and then factorize.*

$$P^{\mu}P_{\mu} - m^2c^4 = 0$$

- Consider only time part $(p_{i=0})$

$$\left(P^0 P_0 - m^2c^2\right) = \left(P^0 + mc\right)\left(P_0 - mc\right) = 0$$

- Two first order equations

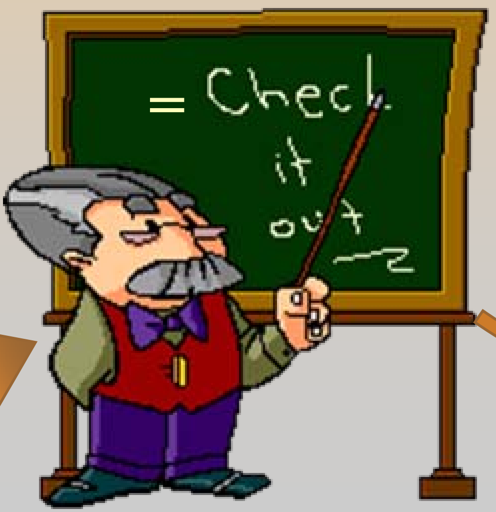
$$\left(P^0 - mc\right) = 0 \quad or \quad \left(P^0 + mc\right) = 0$$

- Either of these guarantees

★  *will happen if we include three (momentum) components of P^μ as well*

$$P^\mu P_\mu - m^2c^2 = \left(\beta^k p_k + mc\right)\left(\gamma^\lambda p_\lambda - mc\right)$$

8 co-efficient



- *Multiplying out RHS*

$$\beta^k p_k \gamma^\lambda p_\lambda - mc\left(\beta^k p_k - \gamma^\lambda p_\lambda\right) - m^2c^2$$

- *We don't want any term linear in* **p**. *Choose* $\beta^k = \gamma^\lambda$

$$\Rightarrow P^\mu P_\mu = \gamma^k \gamma^\lambda p_k p_\lambda$$

- *Find coefficient* γ^λ

$$\left(P^0\right)^2 - \left(P^1\right)^2 - \left(P^2\right)^2 - \left(P^3\right)^2 =$$

$$\gamma^0\gamma^\lambda p_0 p_\lambda + \gamma^1\gamma^\lambda p_1 p_\lambda + \gamma^2\gamma^\lambda p_2 p_\lambda + \gamma^3\gamma^\lambda p_3 p_\lambda$$

- *Further simplification*

$$\left(P^0\right)^2-\left(P^1\right)^2-\left(P^2\right)^2-\left(P^3\right)^2=$$

$$=\left(\gamma^0\right)^2\left(p_0\right)^2+\left(\gamma^1\right)^2\left(p_1\right)^2+\left(\gamma^2\right)^2\left(p_2\right)^2+\left(\gamma^3\right)^2\left(p_3\right)^2$$

$$+\left(\gamma^0\gamma^1+\gamma^1\gamma^0\right)p_0p_1+\left(\gamma^0\gamma^2+\gamma^2\gamma^0\right)p_0p_2$$

$$+\left(\gamma^0\gamma^3+\gamma^3\gamma^0\right)p_0p_3+\left(\gamma^1\gamma^2+\gamma^2\gamma^1\right)p_1p_2$$

$$+\left(\gamma^1\gamma^3+\gamma^3\gamma^1\right)p_1p_3+\left(\gamma^2\gamma^3+\gamma^3\gamma^2\right)p_2p_3$$

- One could pick $\gamma^0 = 1$ and $\gamma^1 = \gamma^2 = \gamma^3 = i$

$$\Rightarrow (\gamma^0)^2 = (1)^2 = 1 \qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = (i)^2 = -1$$

to get rid of the cross-terms $(\gamma^i\gamma^j + \gamma^j\gamma^i)$

Brilliant idea

γ

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 0 \quad \forall \quad i \neq j$$

Matrices

Anticommutator

$$\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\} = 2g^{\mu\nu}$$

Do not commute

Minkowski metric

{1,-1,-1,-1,-1}

★ Define γ matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

0,1 and σ^i are 2×2 matrices

γ's are 4×4 matrices

As a 4×4 matrix equation, the relativistic energy momentum equation can be easily factorize as

$$\left(P^\mu P_\mu - m^2c^2\right) = \left(\beta^k p_k + mc\right)\left(\gamma^\lambda p_\lambda - mc\right) = 0$$

Dirac

$$\left(\gamma^{\lambda} p_{\lambda} - mc\right) = 0$$

Make the usual substitution

$$i\hbar\partial_{\lambda}$$

$$\left(i\hbar\gamma^{\lambda}\,\partial_{\lambda} - mc\right) = 0$$

Let the result act on wave function

know ?

$$\left(i\hbar\gamma^{\lambda}\,\partial_{\lambda} - mc\right)\Psi = 0$$

★ *Ψ must be 4-element column matrix*

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Dirac Spinor /bispinor

Ψ carries 4-components

But

Not 4-vectors

Does not transform under ordinary Lorentz transformation

How does Ψ transform

Bilinear Covariants

- *If we go to a system moving with speed "v" in the x-direction, the transformation rule is*

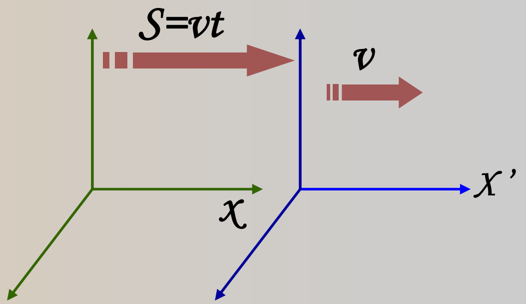
$$\Psi \rightarrow \Psi' = S\,\Psi$$

$$S = \begin{pmatrix} a_+ 1 & a_- \sigma_1 \\ a_- \sigma_1 & a_+ 1 \end{pmatrix}$$

4×4 *matrix*

$$a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Check it! *Its your first assignment*

- *Construct* ***scalar quantity*** *out of spinor Ψ*

$$\Psi^{+}\Psi = \begin{pmatrix}\psi_1^* & \psi_2^* & \psi_3^* & \psi_3^*\end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4\end{pmatrix}$$

Not scalar

$$= |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

- *Appling transformation rule*

$$\widetilde{(AB)}_{ij} = (AB)_{ji} = A_{jk}B_{ki} = \widetilde{B}_{ik}\widetilde{A}_{kj} = \left(\widetilde{B}\widetilde{A}\right)_{ij}$$

$$(AB)^{+} = B^{+}A^{+}$$

* Applying the transformation on bilinear as

$$\left(\Psi^{+}\Psi\right)' = \left(\Psi'\right)^{+}\Psi'$$

$\Psi' = S\,\Psi$

$$\left(S\,\Psi\right)^{+}\left(S\,\Psi\right)$$

$$\Psi^{+}\underbrace{S^{+}S}\,\Psi$$

$$S^{2} = \gamma\begin{pmatrix} 1 & -\frac{v}{c}\sigma_{1} \\ -\frac{v}{c}\sigma_{1} & 1 \end{pmatrix} \neq 1$$

What to do?

- *Introduce the adjoint spinor*

$$\overline{\Psi} = \Psi^{+}\gamma^{0} = \left(\Psi_1^{*}\ \Psi_2^{*}\ -\Psi_3^{*}\ -\Psi_4^{*}\right)$$

- *Now construct relativistic invariant quantity as*

$$\overline{\Psi}\Psi = \Psi^{+}\gamma^{0}\Psi = \left|\Psi_1\right|^2 + \left|\Psi_2\right|^2 - \left|\Psi_3\right|^2 - \left|\Psi_4\right|^2$$

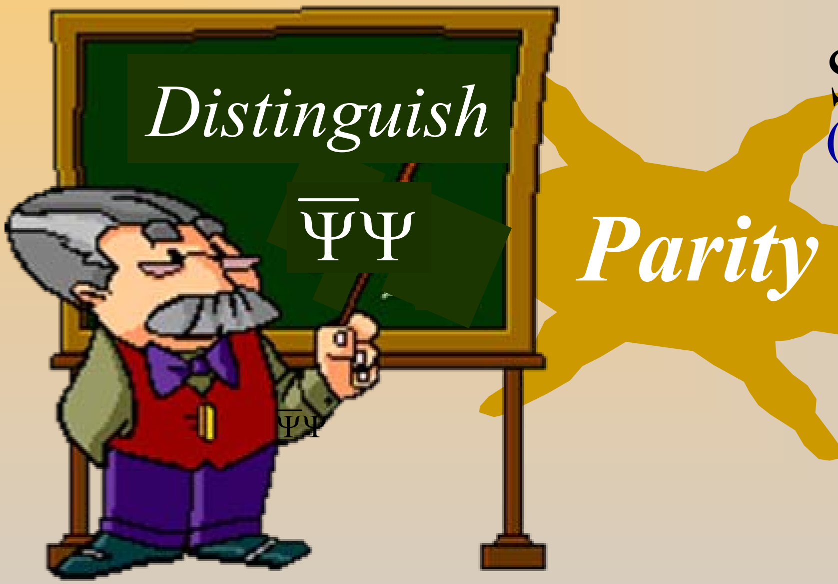
- *Study transformation*

$$\left(\overline{\Psi}\Psi\right)' = \left(\Psi'\right)^{+}\gamma^{0}\Psi' = \left(S\Psi\right)^{+}\gamma^{0}\left(S\Psi\right)$$

$$= \Psi^{+}\underbrace{S^{+}\gamma^{0}S}_{\gamma^{0}}\Psi = \Psi^{+}\gamma^{0}\Psi = \overline{\Psi}\Psi$$

Relativistically Lorentz invariant

Nature of $\bar{\Psi}\Psi$



Scalar
(Does not change sign under parity operator)

$$\hat{P}(x,y,z)\rightarrow(-x,-y,-z)$$

Pseudoscalar
(Change sign under parity operator)

Need to know how Ψ transforms under parity?

Parity transformation

- ***Under parity operation Dirac spinor transform as***

$$\Psi \xrightarrow{\hat{P}} \Psi' = \gamma^0 \Psi$$

$$\left(\overline{\Psi}\Psi\right) \xrightarrow{\hat{P}} \left(\overline{\Psi}\Psi\right)' = \left(\Psi^+ \gamma^0 \Psi\right)' = \Psi^{+'} \gamma^{0'} \Psi'$$

$$= \left(\gamma^0 \Psi\right)^+ \gamma^0 \left(\gamma^0 \Psi\right)$$

$$= \Psi^+ \gamma^{0^+} \gamma^0 \gamma^0 \Psi$$

$$= \Psi^+ \gamma^0 \Psi$$

$$= \left(\overline{\Psi}\Psi\right)$$

invariant under parity

True Scalar

- *One can also make pseudoscalar out of Ψ*

$$\left(\overline{\Psi}\gamma^5\Psi\right)\xrightarrow{\hat{P}}\left(\overline{\Psi}\gamma^5\Psi\right)' = \left(\Psi^+\gamma^0\right)'\gamma^{5\prime}\Psi'$$

$$\Psi \xrightarrow{\hat{P}} \Psi' = \gamma^0\Psi$$

$$= \left(\gamma^0\Psi\right)^+\gamma^0\gamma^5\left(\gamma^0\Psi\right)$$

$$= \Psi^+\gamma^{0^+}\gamma^0\gamma^5\gamma^0\Psi$$

$$= \Psi^+\gamma^5\gamma^0\Psi \quad = -\Psi^+\gamma^0\gamma^5\Psi$$

$$= -\left(\overline{\Psi}\gamma^5\Psi\right)$$

psudoscalar

change sign under parity

Check it! *Its your 2nd assignment*

- ***Check the nature of following quantities under parity transformation?***

1. $\left(\overline{\Psi}\gamma^{\mu}\Psi\right)$
2. $\left(\overline{\Psi}\gamma^{\mu}\gamma^{5}\Psi\right)$
3. $\left(\overline{\Psi}\sigma^{\mu\nu}\Psi\right)$

where

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}\right)$$

Now we know

$$\overline{\Psi} = \Psi^{+}\gamma^{0}$$

$$\Psi \xrightarrow{\hat{P}} \Psi' = \gamma^{0}\Psi$$

$$\left(i\hbar\gamma^{\lambda}\,\partial_{\lambda} - mc\right)\Psi = 0$$

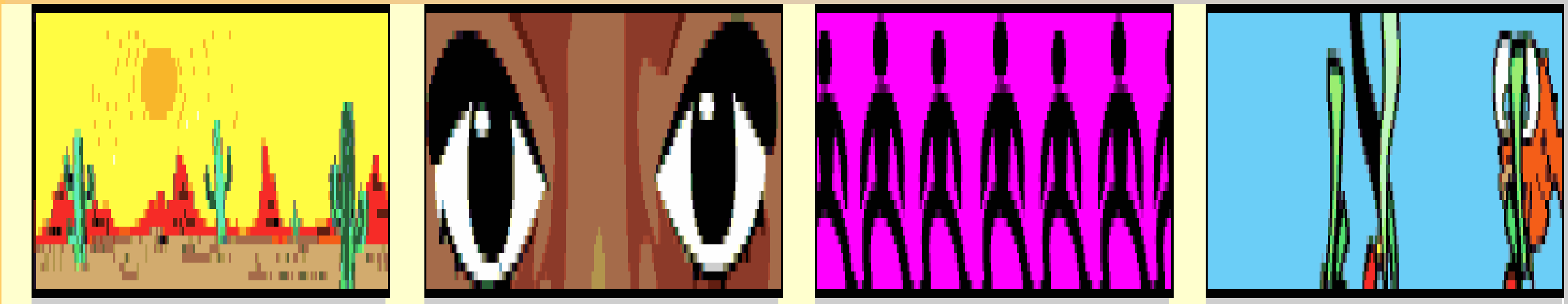
Hurrah!

Ψ

No

Completely know

Ψ



We only know Ψ superficially

What we need to know is

- Detailed structure of Ψ

For this we have to solve Dirac equations

Solution to the Dirac equation

- We know

$$\left(i\hbar\gamma^{\lambda}\,\partial_{\lambda} - mc\right)\Psi = 0 \qquad (1)$$

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

4 element column matrix

- For simple solution, assume Ψ is independent of position

$$\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial y} = \frac{\partial\Psi}{\partial z} = 0$$

- State with zero momentum as

$$p_{\mu} = i\hbar\partial_{\mu}; \quad \mu = 0,1,2,3$$

$$p_0 = i\hbar\partial_0;$$

$$\frac{i\hbar}{c}\gamma^{0}\frac{\partial\Psi}{\partial t} - mc\Psi = 0$$

* simplify

$$\gamma^0 \frac{\partial \Psi}{\partial t} = -i \frac{mc^2}{\hbar} \Psi$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_A}{\partial t} \\ \frac{\partial \psi_B}{\partial t} \end{pmatrix} = -i \frac{mc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

Upper 2 components

$$\psi_A \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$\psi_A \equiv \psi_L$

$$\psi_B \equiv \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

$\psi_B \equiv \psi_R$

Lower 2 components

$$\frac{d\psi_A}{dt} = -i\frac{mc^2}{\hbar}\psi_A$$

$$\frac{d\psi_A}{\psi_A} = -i\frac{mc^2}{\hbar}dt$$

$$\psi_A(t) = e^{-i\frac{mc^2}{\hbar}t}\psi_A(0) = e^{-i\frac{E}{\hbar}t}\psi_A(0)$$

$E = mc^2$

solutions

$E = -mc^2$

$$\frac{d\psi_B}{dt} = i\frac{mc^2}{\hbar}\psi_B$$

$$\frac{d\psi_B}{\psi_B} = i\frac{mc^2}{\hbar}dt$$

$$\psi_B(t) = e^{+i\frac{mc^2}{\hbar}t}\psi_B(0) = e^{i\frac{E}{\hbar}t}\psi_B(0)$$

Closer look of solutions

$$e^{-i\frac{E}{\hbar}t}$$

Characteristic time dependence of quantum state with energy E. For a particle at rest, $E = mc^2$

Ψ_A is exactly what we should have expected if $p = 0$

$$e^{i\frac{E}{\hbar}t} \; ?$$

Represents state with –ev energy

$$E = -mc^2$$

Antiparticle with +ev energy

Ψ_A= electron & Ψ_B= positron

What we learn

- *Dirac equation with* ***p = 0*** *admits four independent solutions*

$$\psi_1 = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

particle

$$\psi_2 = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

particle

$$\psi_3 = e^{i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Anti-part

$$\psi_4 = e^{i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Anti-part

Good enough

What's next

- Plane wave solution when $p \neq 0$

$$\psi(r,t) = a e^{\frac{-i}{\hbar}(Et - p.r)} u(E,p)$$

Bispinor

- In 4-vector notation

$$\psi(X) = a e^{\frac{-i}{\hbar}(X.P)} u(P)$$

Satisfied Dirac eq

- Consider Dirac equation

$$\left(i\hbar\gamma^{\lambda}\,\partial_{\lambda} - mc\right)\Psi(X) = 0$$

$$\Rightarrow \left(i\hbar\gamma^{\lambda}\,\partial_{\lambda} - mc\right) a e^{\frac{-i}{\hbar}(X.P)} u(P) = 0$$

- *x dependence is confined to the exponent*

$$\partial_\lambda \psi(X) = a\, \partial_\lambda \left(e^{\frac{-i}{\hbar}\left(X^\lambda P_\lambda\right)} \right) u(P)$$

$$\partial_\lambda \psi(X) = \frac{-i}{\hbar} P_\lambda a\, e^{\frac{-i}{\hbar}\left(X^\lambda P_\lambda\right)} u(P)$$

Momentum space Dirac equation

$$\left(\gamma^\lambda\ P_\lambda - mc\right) u(P) = 0$$

focus

$$\gamma^{\lambda} P_{\lambda} = \gamma^{0} P_{0} - \gamma^{i} P_{i} = \frac{E}{c}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - p.\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$$

- *Momentum space Dirac equation* *can be written as:*

$$\left(\gamma^{\lambda} P_{\lambda} - mc\right) u = \begin{pmatrix} \left(\frac{E}{c} - mc\right) & -p.\sigma \\ p.\sigma & \left(-\frac{E}{c} - mc\right) \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

Upper 2 components (u_A)

Lower 2 components (u_B)

- In order to satisfy momentum space Dirac eq. we must have

$$\left(\frac{E}{c}-mc\right)u_A - p.\sigma\, u_B = 0$$

$$u_A = \frac{c}{E-mc^2}\left(p.\sigma\right)u_B$$

$$\left(\frac{E}{c}+mc\right)u_B + p.\sigma\, u_A = 0$$

$$u_B = \frac{c}{E+mc^2}\left(p.\sigma\right)u_A$$

$$u_A = \frac{c^2}{E^2-m^2c^4}\left(p.\sigma\right)^2 u_A$$

focus

Calculate $(p.\sigma)^2$

$$p.\sigma = p_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + p_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + p_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$p.\sigma = \begin{pmatrix}p_z & (p_x - ip_y)\\ (p_x + ip_y) & -p_z\end{pmatrix}$$

$$(p.\sigma)^2 = \begin{pmatrix}p_x^2 + p_y^2 + p_z^2 & 0\\ 0 & p_x^2 + p_y^2 + p_z^2\end{pmatrix} = p^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$u_A = \frac{p^2c^2}{E^2 - m^2c^4}u_A$$

Guess what do you get !

Fascinating fact

$$\frac{p^2c^2}{E^2 - m^2c^4} = 1$$

$$E^2 - m^2c^4 = p^2c^2$$

Wow !

Particle state

$$E^2 = \pm\sqrt{m^2c^4 + p^2c^2}$$

Anti-Particle state

Relativistic energy momentum relation enforces by Dirac equation

Construct

4-independent solution to Dirac equation

Evidently particle state

★ *4-independent solution to Dirac equation*

1. *Pick* $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$u_B = \frac{c}{E + mc^2}(p.\sigma)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_B = \frac{c}{E + mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_B = \frac{c}{E + mc^2}\begin{pmatrix} p_z \\ p_x + ip_x \end{pmatrix}$$

E must + ev

otherwise $p \to 0 \Rightarrow u_B \to \infty$

Evidently particle state

★ *4-independent solution to Dirac equation*

2. *Pick* $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_B = \frac{c}{E + mc^2}(p.\sigma)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_B = \frac{c}{E + mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_x & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_B = \frac{c}{E + mc^2}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

E must + ev

otherwise $p \rightarrow 0 \Rightarrow u_B \rightarrow \infty$

Evidently Antiparticle state

★ *4-independent solution to Dirac equation*

3. *Pick* $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$u_A = \frac{c}{E - mc^2}(p.\sigma)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_A = \frac{c}{E - mc^2}\begin{pmatrix} p_z & p_x - ip_x \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_A = \frac{c}{E - mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

E must - ev

otherwise $p \rightarrow 0 \Rightarrow u_A \rightarrow \infty$

Evidently Antiparticle state

★ ***4-independent solution to Dirac equation***

4. Pick $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_A = \frac{c}{E - mc^2}(p.\sigma)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_A = \frac{c}{E - mc^2}\begin{pmatrix} p_z & p_x - ip_x \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_A = \frac{c}{E - mc^2}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

E must - ev

otherwise $p \to 0 \Rightarrow u_A \to \infty$

Summary

- 4-detailed solutions are

$$u_1 = N\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

Pick $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $u_B = \frac{c}{E + mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$

1

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

$E = \sqrt{m^2c^4 + p^2c^2}$

2

$$u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

3

$$u_3 = N\begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x + ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix}$$

4

$$u_4 = N\begin{pmatrix} \frac{c(p_x - ip_y)}{E-mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

$$E=\sqrt{m^2c^4+p^2c^2}$$

$$E=-\sqrt{m^2c^4+p^2c^2}$$

What **is your Opinion ?**

- ***Do*** $\boldsymbol{u_1}$ *and* $\boldsymbol{u_2}$ *describe an electron with spin up and down?*
- ***Do*** $\boldsymbol{u_3}$ *and* $\boldsymbol{u_4}$ *describe positron with spin up and down*

Unfortunately this is not the case!!!

Why

Because

- u_1, u_2, u_3 and u_4 are not the eigenstate of **Dirac particles spin**.

- **Dirac** particles **spin matrices** are

$$S = \frac{\hbar}{2}\Sigma$$

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

Check

- **e.g.** u_1 is not eigenstate of Σ_z $\left(S_z u_1 \neq u_1\right)$

$$S_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$


To do
How to deal with this problem

Very simple Just to

- *Orient the z-axis so that it points along the direction of motion ($p_x = p_y = 0$)*

Then in this case

- $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$ & $\boldsymbol{u_4}$ *are eigenspinor of* $\boldsymbol{S_z}$
- $\boldsymbol{u_1}$ & $\boldsymbol{u_3}$ *are spin up*
- $\boldsymbol{u_2}$ & $\boldsymbol{u_4}$ *are spin down*

Interpretation problem

1

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

✓

2

$$u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

✓

3

$$u_3 = N\begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix}$$

✗

4

$$u_4 = N\begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

✗

Why?

- *But we know all **free particles** must be **alike** and carry **+ev energy** (whether these are positron or electron).*

Solution

- ***-ev energy** solution must be **reinterpreted as +ev energy antiparticle** states*
  - *Flip the sign of E & p in antiparticle state*
- *And get same solution to Dirac equation, with correct physical interpretation*

* *Two spin states of positron with energy E and momentum p*

$$v_1(E,p) \equiv u_4(-E,-p) = N\begin{pmatrix} \frac{c(p_x - ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

$$v_2(E,p) \equiv -u_3(-E,-p) = -N\begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x + ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

Two spin states of positron with energy E & momentum p

with

$$E = \sqrt{m^2c^4 + p^2c^2}$$

4-solutions with correct interpretation

1

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

2

$$u_2 = N\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ \dfrac{-cp_z}{E+mc^2} \end{pmatrix}$$

3

$$v_1(E,p) = N\begin{pmatrix} \dfrac{c(p_x-ip_y)}{E+mc^2} \\ \dfrac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

4

$$v_2(E,p) = -N\begin{pmatrix} \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

Oh No! am I missing something

Ψ

Got it!!!

Ψ

4-solutions with correct interpretation

1

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

2

$$u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

3

$$v_1(E,p) = N \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

4

$$v_2(E,p) = -N \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

Normalization constant N

Wait don't run, try it and you can do it yourself by using following

$$u = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad u^{+} = \begin{pmatrix} \alpha^{*} & \beta^{*} & \lambda^{*} & \delta^{*} \end{pmatrix}$$

$$u^{+}u = \begin{pmatrix} |\alpha|^{2} & |\beta|^{2} & |\lambda|^{2} & |\delta|^{2} \end{pmatrix}$$

Conclusion

$$\left(\gamma^{\lambda} P_{\lambda} - mc\right)u = 0$$ → ***Particle***

$$\left(\gamma^{\lambda} P_{\lambda} + mc\right)v = 0$$ → ***Anti-Particle***

The Quintessence

The Feynman rules for QED

★ **Electron (e^-)**		★ **Positron (e^+)**
$\psi(X) = a e^{\frac{-i}{\hbar}(X.P)} u^s(P)$	**Free**	$\psi(X) = a e^{\frac{i}{\hbar}(X.P)} v(P)$
$(\gamma^\lambda P_\lambda - mc)u = 0$	**Dirac**	$(\gamma^\lambda P_\lambda + mc)v = 0$
$\bar{u}(\gamma^\lambda P_\lambda - mc) = 0$	**Adjoint**	$\bar{v}(\gamma^\lambda P_\lambda + mc) = 0$
$\bar{u}^1 u^2 = 0$	**Orthogonal**	$\bar{v}^1 v^2 = 0$
$\bar{u}u = 2mc$	**Normalized**	$\bar{v}v = -2mc$
$\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\lambda p_\lambda + mc)$	**Completeness**	$\sum_{s=1,2} v^s \bar{v}^s = (\gamma^\lambda p_\lambda - mc)$

Hey! We are done with Dirac equation

Questions are guaranteed in life; Answers aren't.