Quantum ElectroDynamics II

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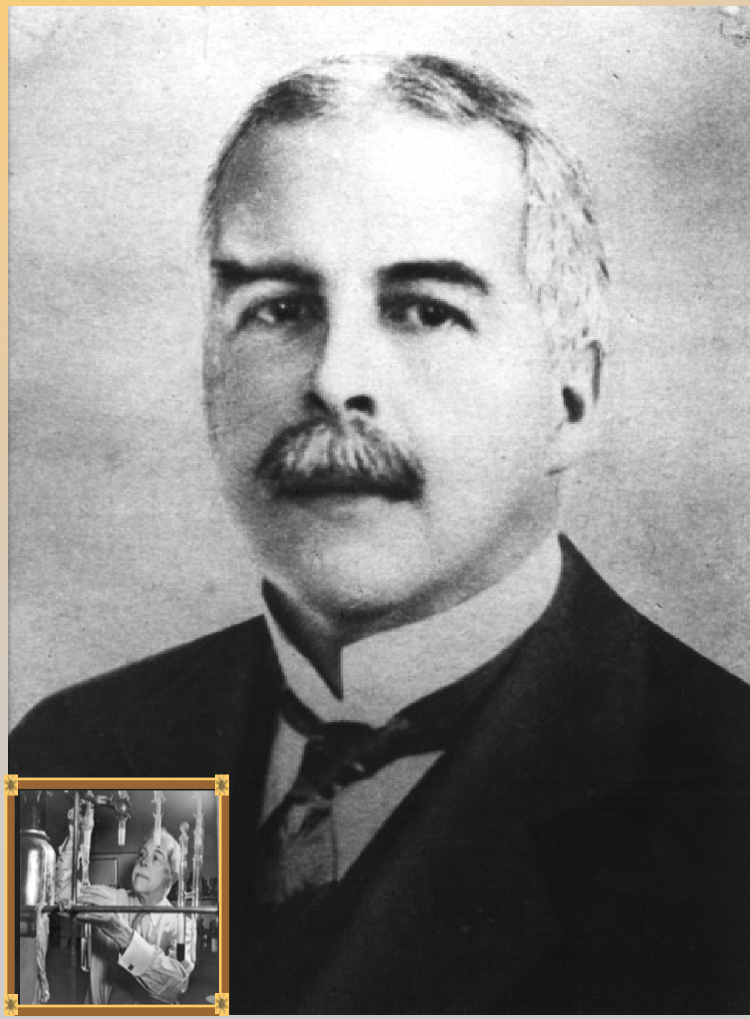
Photon



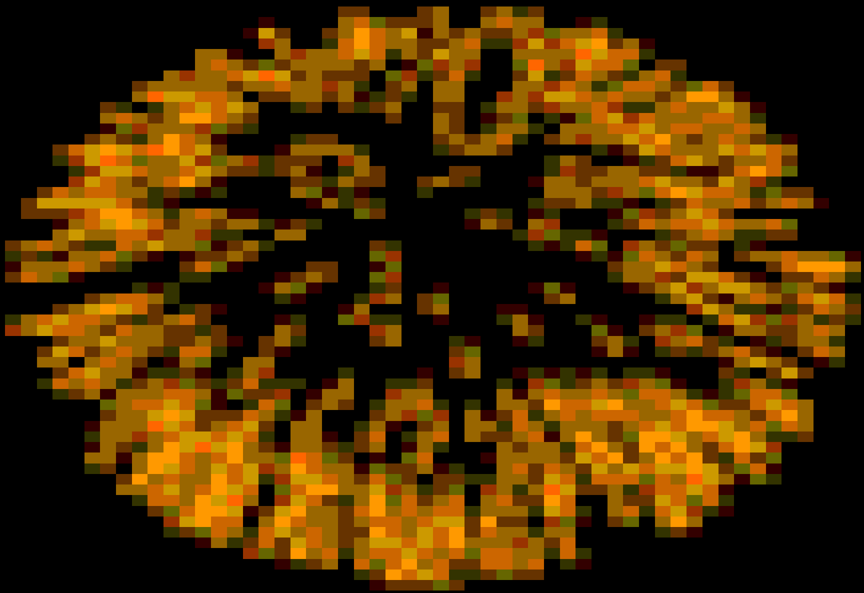
Coined by Gilbert Lewis in 1926.
In Greek Language "Phos" meaning light

The Photons

What do you know about Photon?

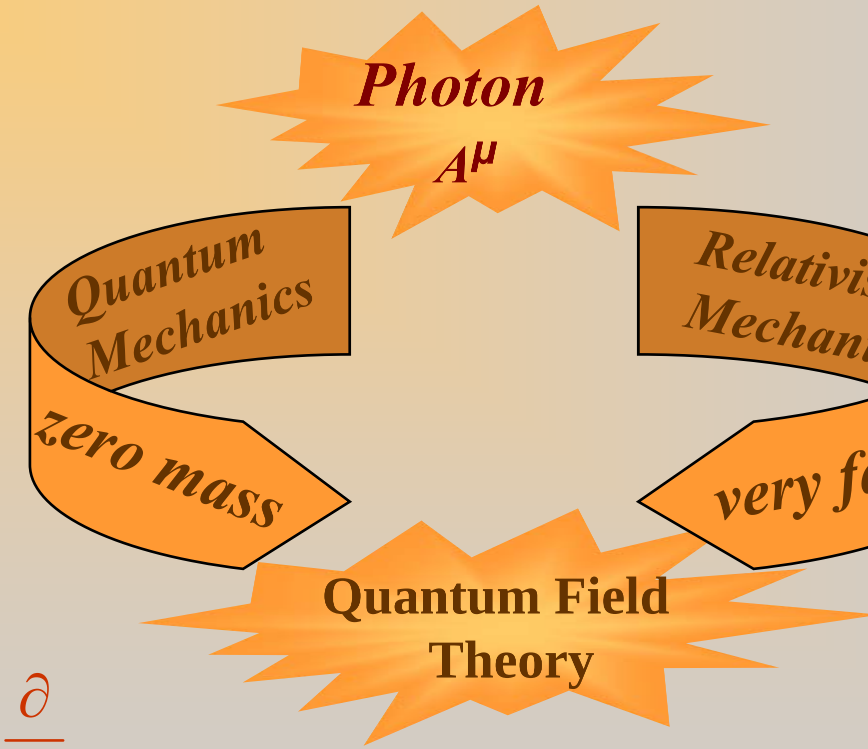
Photon

Discrete bundle (or quantum) of electromagnetic (or light) energy.



Massless spin 1 particle & behaves like both wave and particle

Frame work



$$p = \frac{h}{\lambda}$$

$$E^2 - p^2c^2 = 0$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$P^\mu = (E, p)$$

$$p \rightarrow -i\hbar\nabla = -i\hbar \frac{\partial}{\partial x}$$

$$P^\mu = i\hbar\partial^\mu$$

$$\frac{\partial}{\partial x_\mu}$$

Results due to slight Modification

- *Relativistic Energy-momentum relation for massless particle*

$$E^2 - p^2c^2 = 0$$

- *In four vector notation*

$$P^\mu P_\mu = 0$$

$$\left(-\hbar^2 \partial^\mu \partial_\mu\right) A^\mu = 0$$

$$\Box A^\mu = 0$$

- *Solution is*

$$A^\mu(X) = a e^{-\frac{i}{\hbar} p.x} \varepsilon^\mu(p)$$

$i\hbar\partial^\mu$

$\frac{\partial}{\partial x_\mu}$

$\partial^\mu \partial_\mu$

Polarization vector 4-comp., but not all independent

Electromagnetic Waves

Maxwell's equation

Quick Review

- *Unified description of electricity and magnetism (1864)*

$$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi\, J$$

$$\nabla . E = 4\pi\, \rho$$

inhomogeneous

$$\nabla . B = 0$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

homogeneous

Quick Review

- *Charge conservation comes from the continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla . J = 0$$

Prove by using S.E

$$\nabla . J = -\frac{\partial \rho}{\partial t}$$

4-vector notation

$$J^{\mu} = (\rho, J); \quad \mu = 0,1,2,3$$

Lorentz invariant form

$$\partial_{\mu} J^{\mu} = 0$$

$$\left(\partial_0 J^0 + \partial_i J^i = 0\right)$$

Local charge conservation

Quick Review

- From **homogeneous Maxwell** equation one can get scalar and vector potential (φ, A or A^μ)

$$B = \nabla \times A \quad \text{and} \quad E = -\nabla\phi - \frac{\partial A}{\partial t}$$

- Maxwell equation remains satisfied e.g.

$$B = \nabla.(\nabla \times A) = 0$$

$$\nabla \times E = \nabla \times \left(\nabla\phi - \frac{\partial A}{\partial t}\right)$$

$$\nabla \times E = \nabla \times \nabla\phi - \nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times E = 0 - \frac{\partial}{\partial t}(\nabla \times A)$$

B

Nothing new

Why

To Introduce scalar(φ) and vector (A) potential

- *Just for the sake of convenient mathematical inventions.*

or

- *Due to some concrete reason*

if yes!

Then what that reason is?

$$\phi = \phi' = \phi + \frac{\partial \chi}{\partial t}$$

$$A = A' = A - \nabla \chi$$

defect

arbitrariness

$\chi = \chi(t, x)$

Peruse *the definition of these* ***potentials***

Maxwell equations still satisfied

Change of potentials has no effect on the field

Gauge transformation

Verification

- ***consider***

$A = A' = A - \nabla\chi$

$B = \nabla \times A$

$$\nabla . B' = \nabla . (\nabla \times A')$$

$$\nabla . B' = \nabla . (\nabla \times (A - \nabla\chi))$$

$$\nabla . B' = \nabla . (\nabla \times A) - \nabla . (\nabla \times \nabla\chi)$$

$$\nabla . B' = \nabla . B - 0$$

$$\nabla . B' = 0$$

Magnetic field remains ***invariant*** *under the (local gauge)* ***transformation***

Assignment

$$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi \, J$$

$$\nabla . E = 4\pi \, \rho$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

Remains invariant under (local gauge) transformation?

(gauge freedom)

Enjoy gauge freedom

Exploited it and benefit from it

how

Covariant form of Maxwell's Eq

Relativistically E and B can be represented by antisymmetric 2nd rank tensor, the "field strength tensor," $F^{\mu\nu}$

$$\underset{\mu=0,1,2,3}{F^{\mu\nu}} \equiv \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

In the form of Potentials

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$A^\mu = (\phi, A)$$

Exploit gauge freedom to impose constraint on potential $\partial^\mu A_\mu = 0$

Lorentz condition

Elegant form

- Covariant form of Maxwell eqs .

$$\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \nabla\right)$$

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} \equiv 0$$

$\lambda, \mu, \nu = 0,1,2,3$

Homogeneous

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$J^\mu = (\rho, J)$

In Homogeneous

Verify

$\nabla . E = \rho$

if $\nu = 0$

$\partial_\mu F^{\mu 0} = J^0$

$\mu = 0,1,2,3$

- In terms of 4-vector potential .

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = J^\nu$$

0

$$\Box A^\nu = J^\nu$$

- For free photon (empty space) $J^{\mu} = 0$

$A^0 = 0, \nabla . A = 0$

Coulomb gauge

KG eqs. for massless particle

$$\Box A^{\nu} = 0$$

$$A^{\mu}(X) = a e^{-\frac{i}{\hbar} p.x} \varepsilon^{\mu}(p)$$

Polarization vector 4-comp.,but not all independent

$$P^{\mu} P_{\mu} = 0$$

- Lorentz condition requires that

$$P^{\mu} \varepsilon_{\mu} = 0$$

Information from Coulomb gauge

- *In coulomb gauge*

$$\varepsilon^0 = 0, \quad \varepsilon.p = 0$$

- *Free photon is Transversely polarized*
- *Polarization three vector (ε) is perpendicular to the direction of propagation.*

Coulomb gauge is Transverse gauge

The Feynman rules for QED (γ)

★ Photon (γ)

$$A^{\mu}(X) = a e^{\frac{-i}{\hbar}(X.P)} \varepsilon^{\mu}(s)$$ **Free**

$$\varepsilon^{\mu} P_{\mu} = 0$$ **Lorentz condition**

$$\varepsilon^{\mu *}_{(1)} \varepsilon_{\mu(2)} = 0$$ **Orthogonal**

$$\varepsilon^{\mu *} \varepsilon_{\mu} = 1$$ **Normalized**

$$\sum_{s=1,2} \left(\varepsilon_{(s)}\right)_i \left(\varepsilon_{(s)}\right)_j = \delta_{ij} - \hat{p}_i \hat{p}_j$$ **Completeness**

Lagrangian density

- Lagrangian density for photon field

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F^{\mu\nu} - J^{\mu} A_{\mu}$$

K.E term for γ

photonic field

Externally specified current J^μ is coupled to photon field

Covariant Gauge Transformation

- Gauge transform Lagrangian density

$$\mathcal{L}' = -\frac{1}{4} F^{\mu\nu\prime} F_{\mu\nu}{}' - J^\mu A_\mu{}'$$

using $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

$$= -\frac{1}{4}\left(\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime}\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - J^\mu A_\mu{}'$$

using $A^\mu = A'^\mu = A^\mu - \partial^\mu \chi$ — Gauge T in covariant notation

using $\partial^\mu \partial^\nu \chi = \partial^\nu \partial^\mu \chi$ — Order of diff. is unimport. for scalar

- Simplification after substitutions

$$= -\frac{1}{4}\left(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}\right)\left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right) - J^{\mu} A_{\mu} - J^{\mu} \partial_{\mu} \chi$$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^{\mu} A_{\mu} - J^{\mu} \partial_{\mu} \chi$$

why?

Invariant

Physics remains Invariant

Dirac Equation in electromagnetic field

- *Dirac Equation describing a spin ½ fermions of mass m in free space*

$$\left(i\gamma^{\lambda}\,\partial_{\lambda}-m\right)\Psi=0$$

- *The corresponding Lagrangian density*

$$\mathcal{L}=\overline{\psi}\left(i\gamma^{\lambda}\,\partial_{\lambda}-mc\right)\psi$$

- *What happened to Dirac equation under* ***U(1) gauge*** *transformation i.e.*

$$\Psi(x)\rightarrow\Psi'(x)=e^{-i\alpha}\Psi(x)$$

U(1)

- $n\times n$ *Matrix U is unitary if* $UU^{+}=U^{+}U=1$
- *Product of two unitary Matrix U is unitary.*
- $n\times n$ *unitary Matrices form a group under Matrix multiplication, denoted by U(n).*
- *U(n) has* n^2 *generators.*
- $\det UU^{+}=\det U\det U^{+}=\det U(\det U)^{+}=\det I=1$

$$\det U = e^{-in\alpha}$$

For n=1

$$\det U = e^{-i\alpha}$$

Global gauge transformation

- ***If α is just a number***

$$\mathcal{L} \equiv \overline{\Psi}(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)\Psi(X) \xrightarrow{U(1)} \overline{\Psi}'(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)\Psi'(X) \equiv \mathcal{L}'$$

$$= e^{i\alpha}\overline{\Psi}(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)e^{-i\alpha}\Psi(X)$$

$$= \overline{\Psi}(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)\Psi(X)$$

$$\mathcal{L} \equiv \mathcal{L}'$$

Invariant

Local gauge transformation

- ***If α = α(X)***

$$\mathcal{L} \equiv \overline{\Psi}(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)\Psi(X) \xrightarrow{U(1)} \overline{\Psi}'(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)\Psi'(X) \equiv \mathcal{L}'$$

$$= e^{i\alpha(X)}\overline{\Psi}(X)\left(i\gamma^{\lambda}\,\partial_{\lambda} - m\right)e^{-i\alpha(X)}\Psi(X)$$

$$= e^{i\alpha(X)}\overline{\Psi}(X)\left[i\gamma^{\lambda}\,\partial_{\lambda}\left(e^{-i\alpha(X)}\Psi(X)\right) - m\left(e^{-i\alpha(X)}\Psi(X)\right)\right]$$

$$= e^{i\alpha(X)}\overline{\Psi}(X)\begin{bmatrix} i\gamma^{\lambda}\left\{e^{-i\alpha(X)}\left(\partial_{\lambda}\Psi(X)\right) + \left(\partial_{\lambda}e^{-i\alpha(X)}\right)\Psi(X)\right\} \\ -m\left(e^{-i\alpha(X)}\Psi(X)\right) \end{bmatrix}$$

Local gauge transformation

$$= \overline{\Psi}(X)\left[i\gamma^{\lambda}\left\{\left(\partial_{\lambda}\Psi(X)\right)+\left(\frac{-i\partial\alpha(X)}{\partial X^{\lambda}}\right)\Psi(X)\right\}-m\left(\Psi(X)\right)\right]$$

$$\mathcal{L}=\mathcal{L}+\overline{\Psi}(x)\gamma^{\lambda}\partial_{\lambda}\alpha(X)\Psi(x)$$

$$\mathcal{L}\neq\mathcal{L}'$$

Not Invariant

- *Requirement of local gauge transformation enforces the introduction of electromagnetic field describe by the 4-vector potential*

$$P^\lambda \to P^\lambda - qA^\lambda$$

$$\alpha(X) \equiv q\chi(X)$$

$$\left[\gamma^\lambda\left(i\partial_\lambda - qA_\lambda\right) - m\right]\Psi = 0$$

$$\Psi \to \Psi' = e^{-iq\chi(X)}\Psi$$

$$A_\lambda \to A'_\lambda = A_\lambda - \partial_\lambda \chi$$

* bmn

$$\left[\gamma^{\lambda}\left(i\partial_{\lambda}-qA_{\lambda}\right)-m\right]\Psi \xrightarrow{U(1)} \left[\gamma^{\lambda}\left(i\partial_{\lambda}-qA'_{\lambda}\right)-m\right]\Psi'$$

$$=\left[\gamma^{\lambda}\left\{i\partial_{\lambda}-q\left(A_{\lambda}+\partial_{\lambda}\chi\right)\right\}-m\right]e^{-iq\chi(X)}\Psi=0$$

$$=i\gamma^{\lambda}\partial_{\lambda}\left(e^{-iq\chi(X)}\Psi(X)\right)+\left(-q\gamma^{\lambda}A_{\lambda}-q\gamma^{\lambda}\partial_{\lambda}\chi(X)-m\right)e^{-iq\chi(X)}\Psi(X)=0$$

$$=i\gamma^{\lambda}\left(-iq\partial_{\lambda}\chi(X)\right)\Psi(X)e^{-iq\chi(X)}+i\gamma^{\lambda}e^{-iq\chi(X)}\partial_{\lambda}\Psi(X)$$
$$+\left(-q\gamma^{\lambda}A_{\lambda}-q\gamma^{\lambda}\partial_{\lambda}\chi(X)-m\right)e^{-iq\chi(X)}\Psi(X)=0$$

$$= \left[\cancel{q\gamma^{\lambda}\partial_{\lambda}\chi(X)} + i\gamma^{\lambda}\partial_{\lambda} - q\gamma^{\lambda}A_{\lambda} - \cancel{q\gamma^{\lambda}\partial_{\lambda}\chi(X)} - m \right] e^{-iq\chi(X)}\Psi(X) = 0$$

- After introducing the gauge transformation extra term will exactly cancel out

$$= \left[i\gamma^{\lambda}\partial_{\lambda} - q\gamma^{\lambda}A_{\lambda} - m \right] e^{-iq\chi(X)}\Psi(X) = 0$$

$$\left[\gamma^{\lambda}\left(i\partial_{\lambda} - qA_{\lambda} \right) - m \right]\Psi(X) = 0$$

Complete Lagrangian for f & γ

- Lagrangian density describing the fermionic field in the presence of an electromagnetic field is

$$\mathcal{L} = \overline{\Psi}(X)\left[\gamma^{\lambda}\left(i\partial_{\lambda} - qA_{\lambda}\right) - m\right]\Psi(X) - \frac{1}{4}F^{\lambda\nu}F_{\lambda\nu} - J^{\lambda}A_{\lambda}$$

$$\mathcal{L} = \overline{\Psi}(X)\left[i\gamma^{\lambda}\partial_{\lambda} - m\right]\Psi(X) - \frac{1}{4}F^{\lambda\nu}F_{\lambda\nu} - \left(J^{\lambda} + q\overline{\Psi}(X)\gamma^{\lambda}\Psi(X)\right)A_{\lambda}$$

Current produce by Dirac particle

- Coupling to photon field consist of two parts

    1. With external current density J^{μ} $i.e$ $J^{\lambda}A_{\lambda}$

    2. With fermion field J^{μ} $q\overline{\Psi}(X)\gamma^{\lambda}\Psi(X)$

- *When this current coupled to* A_{λ}*, describe the interaction vertex*