First LHC School Lecture #3

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Gauge Principle

Maxwell's Eqns: in vacuum; away from the sources

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$(\frac{1}{c}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2)\vec{E}, \vec{B} = 0$$

Under space reflection

$$\vec{E} \rightarrow -\vec{E}$$
$$\vec{B} \rightarrow \vec{B}$$

Under time reversal

$$\vec{E} \rightarrow \vec{E}$$
$$\vec{B} \rightarrow -\vec{B}$$

$\vec{E}, \vec{B}$ propagate through space as waves with speed of light c

$$\begin{aligned}
\vec{E} &\sim \vec{\varepsilon} E_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\
\vec{k} &= \text{propagation vector} \\
\omega &= \text{angular frequency} \\
\vec{\nabla}\cdot\vec{E} &= 0 \rightarrow \vec{k}\cdot\vec{\varepsilon} = 0
\end{aligned}$$

i.e. electromagnetic waves are transverse waves; no longitudinal component.

In quantum mechanics, instead of $\vec{E}$ and $\vec{B}$, we deal with vector potential

$$\begin{aligned}
A_\mu &= (A_0, -\vec{A}) = (c\phi, -\vec{A}) \\
A^\mu &= (A^0, \vec{A}) = (c\phi, \vec{A}) \qquad : c = \hbar = 1 \\
\vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \\
\vec{B} &= \vec{\nabla}\times\vec{A}
\end{aligned}$$

$\vec{E}$ and $\vec{B}$ remain unchanged under a transformation (Gauge transformation):

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \partial_\mu \Lambda(x) \\ A_0 &\rightarrow A_0 - \frac{\partial \Lambda}{\partial t} \\ \vec{A} &\rightarrow \vec{A} + \vec{\nabla}\Lambda \end{aligned} \tag{4}$$

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

$$\psi(x,t) \rightarrow \psi'(x,t) = e^{ie\Lambda}\psi(x,t)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + ie\phi \equiv D_t$$

$$\vec{\nabla} \rightarrow \vec{\nabla} - ie\vec{A} \equiv \vec{D}$$

$$i\left(\frac{\partial}{\partial t} + ie\phi\right)\psi = -\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2\psi$$

$$\psi \to e^{ie\Lambda(\vec{x},t)}\psi$$

$$A_\mu \to A_\mu - \partial_\mu \Lambda(x)$$

$$\boldsymbol{L} = -\frac{1}{2m}\vec{\nabla}\psi^* \cdot \vec{\nabla}\psi + \frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$$

$$-e(\rho\phi - \vec{j}\cdot\vec{A}) + \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$\rho = \psi^*\psi,\ j = \frac{1}{2im}\left(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right) - \frac{e}{2m}\vec{A}\psi^*\psi$$

In quantum mechanics, a spin 1/2 particle (electron) is described by a wave function $\psi(x) = \psi(\vec{x}, t)$ which satisfies the Dirac Eq.:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{5}$$

Dirac Eqn. remains invariant under a phase transformation

$$\psi(x) \to e^{ie\Lambda}\psi(x)$$

where Λ is a constant. However if Λ is a function of x, and we still demand that it remains invariant, then it is necessary to introduce a vector field A_μ (vector boson) which is coupled to a vector current with universal coupling e. Such a phase transformation is called the local gauge transformation and the vector boson (quantum of electromagnetic field) associated with the field A_μ is a mediator of force whose strength is determined by e.

Clearly the Dirac Eqn. or the corresponding Lagrangian density

$$L = \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x) \tag{6}$$

is not invariant under the local gauge transformation

$$\psi(x) \to e^{ie\Lambda(x)}\psi(x) \tag{7}$$

In order that the Lagrangian density L be invariant under the gauge transformation (7), we must introduce a vector field $A_\mu(x)$ satisfying Eq.(4) and replace in Eq.(6) $\partial_\mu\psi$ by

$$\partial_\mu\psi(x) \to (\partial_\mu + ieA_\mu)\psi(x) \equiv D_\mu\psi(x) \tag{8}$$

D_μ is called the co-variant derivative. The gauge invariant Lagrangian density is given by

$$L = \bar{\psi}(x)i\gamma^\mu(\partial_\mu + ieA_\mu)\psi(x) - m\bar{\psi}(x)\psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{9}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10}$$

It is easy to see that under the transformation (4), $F_{\mu\nu}$ is invariant. Under the transformations (7) and (4),

$$D_\mu\psi \to e^{ie\Lambda(x)}D_\mu\psi \tag{11}$$

so that $\bar{\psi}D_\mu\psi$ is gauge invariant, and so is $m\bar{\psi}\psi$. From Eq.(6), we see that the interaction of matter field ψ with the electromagnetic field A_μ is given by

$$L_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu = -J^\mu_{em}A_\mu \tag{12}$$

where

$$J^{\mu}_{em} = e\bar{\psi}\gamma^{\mu}\psi \quad , \quad \partial_{\mu} J^{\mu}_{em} = 0 \tag{13}$$

is the electromagnetic current. We conclude that the gauge principle viz. the invariance of a fundamental physical law under the gauge transformation gives correctly the form of interaction of a charged particle with electromagnetic field. To sum up the consequences of the electromagnetic force as a gauge force are as follows:

- It is universal viz. any charged particle is coupled with the electromagnetic field A_{μ} with a universal coupling strength given by e, the electric charge of the particle.

- J^{μ}_{em} is conserved.

- The electromagnetic field is a vector field and hence the associated quantum, the photon, has spin 1,

$$J^P = 1^-$$

no longitudinal polarization

$$\vec{k} \cdot \vec{\varepsilon} = 0$$

- The photon must be massless, since the mass term $\mu^2 A^\mu A_\mu$ is not invariant under the gauge transformation. Thus unbroken gauge symmetry gives rise to long range force mediated by a massless gauge boson i.e. photon.

The covariant derivative D_μ is an operator whose commutator is

$$\begin{aligned} [D_\mu, D_\nu] &= ieF_{\mu\nu} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{14}$$