Black HOles in Non-Commutative Geometry

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Schwarzschild black hole

The Einstein Field Equations

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{\kappa}{c^2} T_{ab} \ ,$$

metric of a Riemann manifold

$$\mathrm{d}s^2 = g_{ab} \mathrm{d}x^a \mathrm{d}x^b \ ,$$

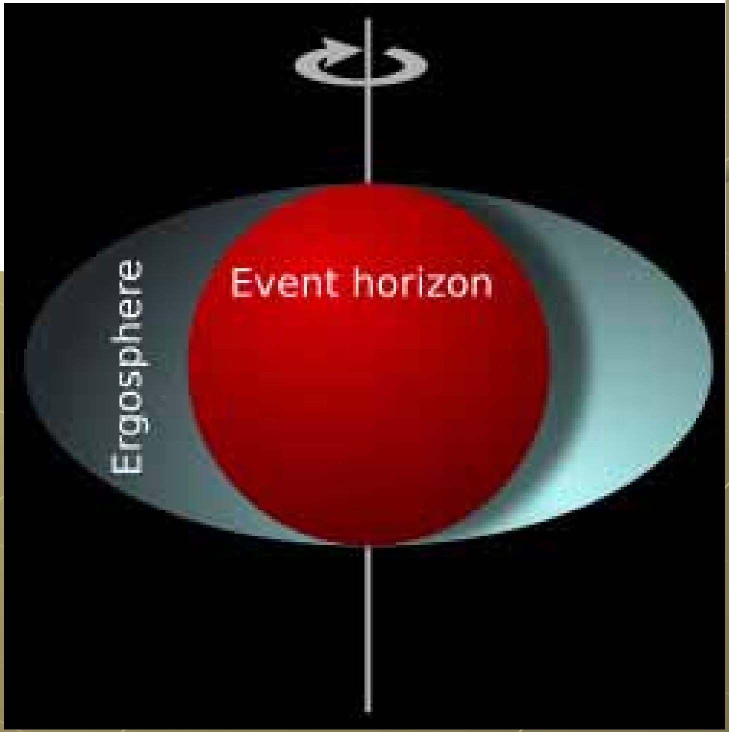
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The Kerr-Newman black hole

$$ds^2 = -\frac{\Delta^2}{\rho^2}(dt - asin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}(adt - (r^2 + a^2)d\phi)^2,$$

where

$$\begin{aligned} \Delta(r)^2 &= (r^2 + a^2) - 2Mr + Q^2, \\ \rho^2(r,\theta) &= r^2 + a^2 cos^2\theta, \\ a &= \frac{J}{M}. \end{aligned}$$



To obtain the horizons for this metric, we put $g^{rr} = 0$, and get

$$r_{\pm} = M + \sqrt{M^2 - a^2 - Q^2}.$$

The Hawking temperature is defined as

$$T = \frac{\hbar}{4\pi} \frac{r_+ - r_-}{r_+^2 + a^2},$$

which in our case takes the form

$$T = \frac{\hbar}{2\pi} \frac{\sqrt{M^4 - J^2 - Q^2 M^2}}{M\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2}\right)}.$$

The angular velocity, $\Omega = a/(r_+^2 + a^2)$, for the above metric takes the form

$$\Omega = \frac{J}{M\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2}\right)},$$

and the electrostatic potential, $\Phi = r_+ Q/(r_+^2 + a^2)$, becomes

$$\Phi = \frac{Q\left(M^2 + \sqrt{M^4 - J^2 - Q^2 M^2}\right)}{M\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2}\right)}.$$

Exact differentials in three variables and the first law of thermodynamics

The three dimensional differential of a function *f*

$$df(x,y,z) = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$$

is exact if the following conditions hold

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}, \quad \frac{\partial A}{\partial z} = \frac{\partial C}{\partial x}, \quad \frac{\partial B}{\partial z} = \frac{\partial C}{\partial y}.$$

Here we have

$$\frac{\partial f}{\partial x} = A, \frac{\partial f}{\partial y} = B, \frac{\partial f}{\partial z} = C.$$

$$
\begin{aligned}
f(x,y,z) = & \int A dx + \int B dy + \int C dz \\
& - \int \left(\frac{\partial}{\partial y} \left(\int A dx \right) \right) dy - \int \left(\frac{\partial}{\partial z} \left(\int A dx \right) \right) dz - \int \left(\frac{\partial}{\partial z} \left(\int B dy \right) \right) dz \\
& + \int \frac{\partial}{\partial z} \left(\int \left(\frac{\partial}{\partial y} \left(\int A dx \right) \right) dy \right) dz.
\end{aligned}
$$

M, J, Q, the mass, angular momentum and charge of the black hole

$$dM = TdS + \Omega dJ + \Phi dQ,$$

T is the temperature, S entropy, Ω angular velocity and Φ electrostatic potential

$$dS(M, J, Q) = \frac{1}{T} dM - \frac{\Omega}{T} dJ - \frac{\Phi}{T} dQ.$$

$$\frac{\partial}{\partial J}\left(\frac{1}{T}\right) = \frac{\partial}{\partial M}\left(-\frac{\Omega}{T}\right)$$
$$\frac{\partial}{\partial Q}\left(\frac{1}{T}\right) = \frac{\partial}{\partial M}\left(-\frac{\Phi}{T}\right)$$
$$\frac{\partial}{\partial Q}\left(-\frac{\Omega}{T}\right) = \frac{\partial}{\partial J}\left(-\frac{\Phi}{T}\right)$$

$$\frac{\partial}{\partial J}\int\left(\frac{dM}{T}\right) = -\frac{\Omega}{T},$$

$$\frac{\partial}{\partial Q}\int\left(\frac{dM}{T}\right) = -\frac{\Phi}{T},$$

$$\frac{\partial}{\partial Q}\int\left(-\frac{\Omega dJ}{T}\right) = -\frac{\Phi}{T}.$$

$$\begin{aligned} S(M,J,Q) &= \int \frac{dM}{T} \\ &= \frac{\pi}{\hbar}\int \frac{M\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2M^2}\right)}{\sqrt{M^4 - J^2 - Q^2M^2}} dM \\ &= \frac{\pi}{\hbar}\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2M^2}\right), \end{aligned}$$

$$S = \frac{\pi}{\hbar}(r_+^2 + a^2).$$

Corrected Hawking temperature is

$$T_c = T\left(1 + \frac{\alpha_i \hbar^i}{(r_+^2 + a^2)^i}\right)^{-1}$$

And the surface gravity becomes

$$\mathcal{K} = 2\pi T,$$

$$\mathcal{K} = \mathcal{K}_0\left(1 + \sum_i \frac{\alpha_i \hbar^i}{(r_+^2 + a^2)^i}\right)^{-1}$$

$$\frac{\partial}{\partial J}\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)=\frac{\partial}{\partial M}\sum\frac{-\Omega}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right),$$
$$\frac{\partial}{\partial Q}\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)=\frac{\partial}{\partial M}\sum\frac{-\Phi}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right),$$
$$\frac{\partial}{\partial Q}\sum\frac{-\Omega}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)=\frac{\partial}{\partial J}\sum\frac{-\Phi}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right).$$

$$\begin{aligned}
S(M,J,Q) =& \int\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dM-\int\sum\frac{\Omega}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dJ\\
&-\int\sum\frac{\Phi}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dQ\\
&-\int\left(\frac{\partial}{\partial J}\left(\int\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dM\right)\right)dJ\\
&-\int\left(\frac{\partial}{\partial Q}\left(\int\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dM\right)\right)dQ\\
&+\int\left(\frac{\partial}{\partial Q}\left(\int\sum\frac{\Omega}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dJ\right)\right)dQ\\
&+\int\frac{\partial}{\partial Q}\left(\int\left(\frac{\partial}{\partial J}\left(\int\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dM\right)\right)dJ\right)dQ\,.
\end{aligned}$$

$$S(M,J,Q)=\int\sum\frac{1}{T}\left(1+\frac{\alpha_i\hbar^i}{(r_+^2+a^2)^i}\right)dM,$$

$$S(M,J,Q)=\int\frac{1}{T}dM+\int\frac{\alpha_1\hbar}{T(r_+^2+a^2)}dM+\int\frac{\alpha_2\hbar^2}{T(r_+^2+a^2)^2}dM$$
$$+\int\frac{\alpha_3\hbar^3}{T(r_+^2+a^2)^3}dM+\cdots$$
$$=I_1+I_2+I_3+I_4+\cdots,$$

$$I_1=\frac{\pi}{\hbar}(r_+^2+a^2)$$

$$I_2=\pi\alpha_1 ln(r_+^2+a^2).$$

$$I_k=\frac{\pi\alpha_{k-1}\hbar^{k-2}}{(2-k)(r_+^2+a^2)^{k-2}},k>2.$$

$$S=\frac{\pi}{\hbar}(r_+^2+a^2)+\pi\alpha_1 ln(r_+^2+a^2)+\sum_{k>2}\frac{\pi\alpha_{k-1}\hbar^{k-2}}{(2-k)(r_+^2+a^2)^{k-2}}.$$

Bekenstein-Hawking area law

$$S = \frac{A}{4},$$

$$A = 4\pi(r_+^2 + a^2),$$

$$S = \frac{A}{4\hbar} + \pi\alpha_1 lnA - \frac{4\pi^2\alpha_2\hbar}{A} - \frac{8\pi^3\alpha_3\hbar^2}{A^2} - \cdots,$$

Einstein-Maxwell Dilaton-Axion black hole

$$ds^2 = -\frac{\Sigma - a^2 sin^2\theta}{\Delta} dt^2 - \frac{2a sin^2\theta}{\Delta}\left[(r^2 - 2Dr + a^2) - \Sigma\right] dt d\phi$$

$$+\frac{\Delta}{\Sigma} dr^2 + \Delta d\theta^2 + \frac{sin^2\theta}{\Delta}\left[(r^2 - 2Dr + a^2)^2 - \Sigma a^2 sin^2\theta\right] d\phi^2$$

$$\Delta = r^2 - 2Dr + a^2 cos^2\theta$$

$$\Sigma = r^2 - 2Mr + a^2$$

Electric charge

$$Q = \sqrt{2kD(D-M)}, \text{ where } k = e^{2\Phi_0}$$

The vector potential

$$A\mu = -\frac{Qr}{\rho^2}(\delta^0_\mu - asin^2\theta\delta^0_\mu)$$

$$\rho^2 = r^2 + a^2cos^2\theta$$

Horizons

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$$

Outer horizon

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}$$

Temperature associated with this horizon

$$T = \frac{\hbar}{4\pi}\left[\frac{(r_+ - M - D)}{(r_+^2 - 2Dr_+ + a^2)}\right]$$

$$T = \frac{\hbar}{4\pi}\left[\frac{\sqrt{M^2(M+D)^2 - J^2}}{M(M(M+D) + \sqrt{M^2(M+D)^2 - J^2}}\right]$$

Angular velocity on the horizon

$$\Omega = \frac{J/M}{r_+^2 - 2Dr_+ + a^2}$$

$$\Omega = \frac{J}{2M\left[M(M+D) + \sqrt{M^2(M+D)^2 - J^2}\right]}$$

The electrostatic potential

$$\Phi = \frac{-2DM}{Q(r_+^2 - 2Dr_+ + a^2)}$$

$$
\begin{aligned}
S(M,J,Q) &= \int \frac{dM}{T} \\
&= \frac{4\pi}{\hbar} \int \frac{M\left[(M(M+D)+\sqrt{M^2(M+D)^2-J^2}\right]}{\sqrt{M^2(M+D)^2-J^2}} dM \\
&= \frac{2\pi}{\hbar}\left[M(M+D)+\sqrt{M^2(M+D)^2-J^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
S(M,J,Q) &= \int \frac{dM}{T} \\
&= \frac{4\pi}{\hbar} \int \frac{M\left[(M(M+D)+\sqrt{M^2(M+D)^2-J^2}\right]}{\sqrt{M^2(M+D)^2-J^2}} dM \\
&= \frac{2\pi}{\hbar}\left[M(M+D)+\sqrt{M^2(M+D)^2-J^2}\right]
\end{aligned}
$$

$$
\frac{\partial}{\partial J}\left(\frac{1}{T}\right)\left(1+\sum \frac{\alpha_i \hbar^i}{(r_+^2-2Dr_+ + a^2)^i}\right) = \frac{\partial}{\partial M}\left(\frac{-\Omega}{T}\right)\left(1+\sum \frac{\alpha_i \hbar^i}{(r_+^2-2Dr_+ + a^2)^i}\right)
$$

$$
\frac{\partial}{\partial Q}\left(\frac{1}{T}\right)\left(1+\sum \frac{\alpha_i \hbar^i}{(r_+^2-2Dr_+ + a^2)^i}\right) = \frac{\partial}{\partial M}\left(\frac{-\Phi}{T}\right)\left(1+\sum \frac{\alpha_i \hbar^i}{(r_+^2-2Dr_+ + a^2)^i}\right)
$$

$$
\frac{\partial}{\partial Q}\left(\frac{-\Omega}{T}\right)\left(1+\sum \frac{\alpha_i \hbar^i}{(r_+^2-2Dr_+ + a^2)^i}\right) = \frac{\partial}{\partial J}\left(\frac{-\Phi}{T}\right)\left(1+\sum \frac{\alpha_i \hbar^i}{(r_+^2-2Dr_+ + a^2)^i}\right)
$$

$$S(M,J,Q) = \int \frac{1}{T}\left(1 + \sum \frac{\alpha_i \hbar^i}{(r_+^2 - 2Dr_+ + a^2)^i}\right) dM$$

$$\begin{aligned} S(M,J,Q) = & \int \frac{1}{T} dM + \int \frac{\alpha_1 \hbar}{T(r_+^2 - 2Dr_+ + a^2)} dM \\ & + \int \frac{\alpha_2 \hbar^2}{T(r_+^2 - 2Dr_+ + a^2)^2} dM \\ & + \int \frac{\alpha_3 \hbar^3}{T(r_+^2 - 2Dr_+ + a^2)^3} dM + \cdots \\ & = I_1 + I_2 + I_3 + I_4 + \cdots \end{aligned}$$

$$S(M,J,Q) = \int \frac{1}{T}\left(1 + \sum \frac{\alpha_i \hbar^i}{(r_+^2 - 2Dr_+ + a^2)^i}\right) dM$$

$$\begin{aligned} S(M,J,Q) = & \int \frac{1}{T} dM + \int \frac{\alpha_1 \hbar}{T(r_+^2 - 2Dr_+ + a^2)} dM \\ & + \int \frac{\alpha_2 \hbar^2}{T(r_+^2 - 2Dr_+ + a^2)^2} dM \\ & + \int \frac{\alpha_3 \hbar^3}{T(r_+^2 - 2Dr_+ + a^2)^3} dM + \cdots \\ & = I_1 + I_2 + I_3 + I_4 + \cdots \end{aligned}$$

$$I_2 = 2\pi\alpha_1\hbar \int \frac{M dM}{\sqrt{M^2(M+D)^2 - J^2}}$$

$$I_2 = \pi\alpha_1 ln \left| M(M+D) + \sqrt{M^2(M+D)^2 - J^2} \right|,$$

$$I_2 = \pi\alpha_1 ln(r_+^2 - 2Dr_+ + a^2)$$

The k-th integral I_k where $k = 3, 4, \cdots$

$$I_k = \int \frac{\alpha_{k-1}\hbar^{k-1}dM}{T(r_+^2 - 2Dr_+ + a^2)^{k-1}}$$
$$= \frac{2\pi}{\hbar}\int \frac{\alpha_{k-1}\hbar^{k-1}MdM}{\sqrt{M^2(M+D)^2 - J^2}\left[M(M+D) + \sqrt{M^2(M+D)^2 - J^2}\right]^{k-2}}$$
$$= \frac{\pi\alpha_{k-1}\hbar^{k-2}}{2-k}\left[M(M+D) + \sqrt{M^2(M+D)^2 - J^2}\right]^{2-k}, k > 2$$

$$I_k = \frac{\pi\alpha_{k-1}\hbar^{k-2}}{(2-k)(r_+^2 - 2Dr_+ + a^2)^{k-2}}, k > 2$$

Thus the entropy including all the correction terms becomes

$$S = \frac{\pi}{\hbar}(r_+^2 - 2Dr_+ + a^2) + \pi\alpha_1 ln(r_+^2 - 2Dr_+ + a^2)$$
$$+ \sum_{k>2} \frac{\pi\alpha_{k-1}\hbar^{k-2}}{(2-k)(r_+^2 - 2Dr_+ + a^2)^{k-2}} + \cdots$$

Area

$$A = 4\pi(r_+^2 - 2Dr_+ + a^2)$$

$$S = \frac{A}{4}$$

$$S = \frac{A}{4\hbar} + \pi\alpha_1 lnA - \frac{4\pi^2\alpha_2\hbar}{A} - \frac{8\pi^3\alpha_3\hbar^2}{A^2} - \cdots$$

Non-Commutative (NC) geometry

- Classical BH vs. quantum BH
- `Hawking Paradox': temperature diverges as radius goes to zero
- In commutative space the mass density of a point particle is product of mass with Dirac delta function. NC space is fuzzy; Dirac delta function replaced by Gaussian distribution. Mass is not located at a point; it is smeared around a region.
- NC geometry provides mathematical framework for quantum gravity
- Information loss: unitarity violated
- At Planck scales coordinates become NC:
- NC corrections to thermodynamical quantit

$$[x^\mu, x^\nu] = \theta^{\mu\nu}$$

Non-commutative corrections to the Schwarzschild black hole

Gaussian distribution of minimal width $\sqrt{\theta}$,

$$\rho_\theta = \frac{M}{(4\pi\theta)^{3/2}} e^{-(r^2/4\theta)}.$$

$$m_\theta(r) = \int_0^r 4\pi r'^2 \rho_\theta(r') dr' = \frac{2M}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{}\right),$$

where $\gamma(\frac{3}{2}, \frac{r^2}{4\theta})$ is the lower incomplete gamma function

Incomplete gamma function.—The lower incomplete gamma function is given by

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt, \qquad)$$

$$ds^2 = -\left(1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right) dt^2 + \left(1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$T_h = \frac{\mathcal{K}}{2\pi} = \frac{1}{4\pi}\left[\frac{1}{r_h} - \frac{r_h^2}{4\theta^{3/2}}\frac{e^{-(r_h^2/4\theta)}}{\gamma\left(\frac{3}{2}, \frac{r_h^2}{4\theta}\right)}\right]\left(1 + \frac{4\alpha}{r_h^2}\right).$$

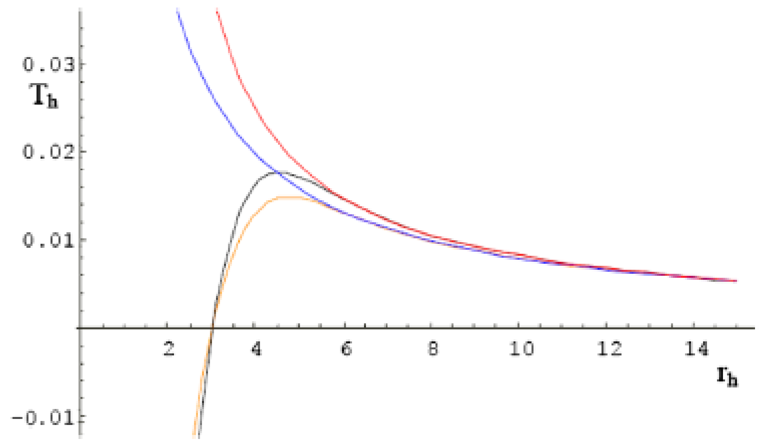


FIG. 1 (color online). T_h Vs. r_h plot (Here $\alpha = \theta$, α, and θ are positive.) r_h is plotted in units of $\sqrt{\theta}$, and T_h is plotted in units of $\frac{1}{\sqrt{\theta}}$. Red curve: $\alpha \neq 0$, $\theta = 0$. Blue curve: $\alpha = 0$, $\theta = 0$. Black curve: $\alpha \neq 0$, $\theta \neq 0$. Yellow curve: $\alpha = 0$, $\theta \neq 0$.

Nicolini, Smailagic, Spallucci, PLB 2006

Banerjee, Majhi, Samanta, PRD 2008

Conclusion

- Bekenstein-Hawking Entropy
- First Law of thermodynamics in 3 parameters
- Exactness of entropy differential
- General framework
- Charged Kerr black hole
- Einstein-Maxwell dilaton-axion black hole
- NC Schwarzschild