An Excursion into the Anti-deSitter Spacetime and the World of Holography

Abstract

The geometry of the anti-deSitter spacetime, a space with a negative cosmological constant, is examined. The geodesics are shown to be periodic in time, as they must since the whole spacetime is periodic with respect to time. The holographic principle is presented and the validity of the entropy constraint inferred from that principle is checked for our universe and for a universe filled with radiation. The constraint is found to be fulfilled in all cases. Finally, an indication is given of how the holographic principle is realized in the anti-deSitter spacetime.

Contents

1 Introduction

As Einstein realized that his field equations of gravity ruled out a static universe, his conception of the world forced him to introduce the cosmological constant. By this trick the static universe was, more or less, a possible solution to the equations. Soon thereafter, Hubble discovered the redshift of distant stars and the expanding universe was a fact. Since then Einstein talked about the cosmological constant as the mistake of his life.

But regardless of what originally made Einstein introduce this constant, there are good reasons for keeping it in the equations. The most general versions of the equations contain a cosmological constant, and there are so far no good physical arguments for excluding it. Recent measurements show that even though the cosmological constant in the present epoch is extremely tiny, it is probably not identical to zero but measurably larger [1]. Cosmologists believe that, during an epoch shortly after big bang, the cosmological constant was considerable, causing the universe to expand exponentially in time. This is called the epoch of inflation and fragments of evidence exist for this period to be a historical fact [2]. New satellites will in a couple of years permanently confirm or reject this evidence [3].

Even if theory is not able to get rid of the cosmological constant completely, it seems to be able to restrict the sign of it. Hawking and Page have shown that a negative cosmological constant results in instabilities which could hardly be compatible with the present universe [4]. However, the corresponding spacetime, the so called anti-deSitter spacetime, has been subject for deep research during the last years. The physicists have discovered extraordinary properties inherent in the anti-deSitter spacetime, allowing for an interesting connection between string theory and certain ordinary field theories [5][6]. Furthermore, the holographic principle (described in section 3) seems to be built in for theories in the anti-deSitter spacetime, which is generally not the case for other geometries.

This report is an attempt to clarify the basics in the anti-deSitter spacetime and the principle of holography. In section 2 useful metrics are presented and the geometry of the anti-deSitter cylinder is analyzed. Section 3 discusses the underlying concepts for the holographic principle, and section 4 examines the validity of the holographic entropy bounds in our universe, in the past and in the future. The dependence of the result with respect to the matter-radiation composition is also discussed. In section 5 we will get a glimpse of how the holographic principle is fulfilled in the anti-deSitter spacetime, considering the entropy of a big black hole. Finally, a summary with conclusions is found in section 6.

2 Anti-deSitter Spacetime

With respect to the great interest for the anti-deSitter spacetime at the moment, it is helpful for every physicist to have an intuition for the corresponding geometry. Formally, the anti-deSitter spacetime is defined as an empty space solution to the Einstein field equations with a negative cosmological constant. The metric for a d-dimensional anti-deSitter spacetime can be obtained by embedding a $(d + 1)$-dimensional hyperboloid in a flat $(d+1)$-dimensional space with two time-directions. For convenience, only the 3-dimensional case is shown explicitly here (the general metric can be found in e.g. [5]). The equation for the hyperboloid, defining a 3-dimensional surface, may be written

$$-U^2 - V^2 + X^2 + Y^2 = -b^2$$

where U, V, X and Y are defined throughout the real axis, and b is a real constant. To arrive at the anti-deSitter metric, this hyperboloid should be embedded into the metric

$$ds^2 = -\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}X^2 + \mathrm{d}Y^2.$$

There are several convenient ways of parameterizing the hyperboloid. Here, two parameterizations will be presented, leading to the global and the Poincaré coordinates, respectively.

2.1 Global Coordinates

So-called global coordinates are obtained by parameterizing the hyperboloid in the following way

$$\begin{cases} U &=& b \, \cosh\mu \, \sin t \\ V &=& b \, \cosh\mu \, \cos t \\ X &=& b \, \sinh\mu \, \cos\theta \\ Y &=& b \, \sinh\mu \, \sin\theta \end{cases}.$$

Clearly, t and θ are periodic, and we define the range of definition of the variables to be $-\pi \leq t \leq \pi$, $0 \leq \theta \leq 2\pi$ and $\mu \geq 0$. As these variables are substituted into the metric, we get

$$ds^2 = b^2(-\cosh^2\mu \, dt^2 + d\mu^2 + \sinh^2\mu \, d\theta^2).$$

Usually, one furthermore lets $\sinh\mu = \tan\rho$, where $0 \leq \rho \leq \pi/2$, turning the metric into the form

$$ds^2 = b^2(-\sec^2\rho \, dt^2 + \sec^2\rho \, d\rho^2 + \tan^2\rho \, d\theta^2).$$

In the variables (t, ρ, θ) the anti-deSitter spacetime can be viewed as a cylinder, as shown in figure 1. Examples of how geodesic observers and light-rays behave in this cylinder are given in the last parts of this section.

2.2 Poincaré Coordinates

Another set of coordinates describing the same spacetime is called the Poincaré coordinates, and is obtained by letting

$$\left\{\begin{array}{lcl} U & = & (b^2 + x^2 + r^2 - t^2)/2r \\ V & = & b\ t/r \\ X & = & (b^2 - x^2 - r^2 + t^2)/2r \\ Y & = & b\ x/r \end{array}\right..$$

Here t and x may be assigned any real value, and $r \geq 0$. The metric now reads

$$\mathrm{d}s^2 = \frac{b^2}{r^2}(-\mathrm{d}t^2 + \mathrm{d}r^2 + \mathrm{d}x^2).$$

How are the Poincaré coordinates related to the global coordinates? By simply identifying their definitions it is seen that

$$\left\{\begin{array}{rcl} b\ \sin t\ \sec \rho & = & (b^2 + x^2 + r^2 - t'^2)/2r \\ \tan \rho\ \cos \theta & = & x/r \\ b\ \tan \rho\ \sin \theta & = & (b^2 - x^2 - r^2 + t'^2)/2r \\ \cos t\ \sec \rho & = & t'/r \end{array}\right.,$$

where t' denotes the Poincaré time coordinate. By adding the first equation to the third we get

$$\sin t\ \sec \rho + \tan \rho\ \sin \theta = \frac{b}{r}.$$

As far as the the definition ranges for the global coordinates are concerned, the left hand side ranges throughout the real axis. By symmetry it is then clear that half of the points (t, ρ, θ) are excluded in this equation, since the right hand side is always positive. This implies that Poincaré coordinates only cover one half of the anti-deSitter spacetime. The other half is reached by letting $r \leq 0$. The equations for the boundary surfaces between the regions where r is positive and negative are obtained by letting $r \to \infty$, resulting in

$$\sin t + \sin \rho\ \sin \theta = 0.$$

These surfaces are indicated in figure 1. The exact shape of the central surface which resides in the interval $-\pi/2 \leq t \leq \pi/2$ is shown in figure 2.
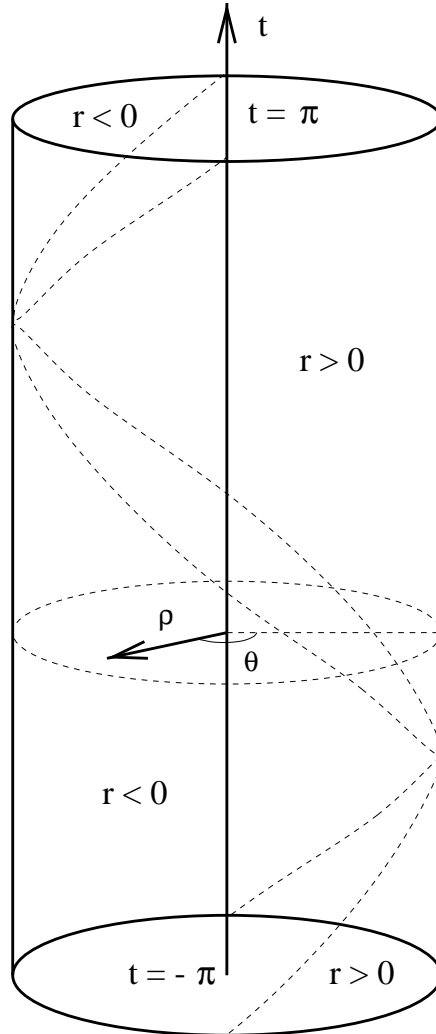
Figure 1: In global coordinates t, ρ and θ, the spacetime can be viewed as a cylinder, the so-called anti-deSitter cylinder. The top and bottom of the cylinder should be identified. The regions for the different Poincaré parameterizations are also shown.

Figure 2: The central surface in the anti-deSitter cylinder which separates the different Poincaré parameterizations of the hyperboloid. The surface is obtained by letting $r \rightarrow \pm\infty$. There are similar surfaces in the intervals $-\pi \leq t \leq -\pi/2$ and $\pi/2 \leq t \leq \pi$, which are mirror images to the shown surface, mirrored in the planes $t = \pm\pi/2$.

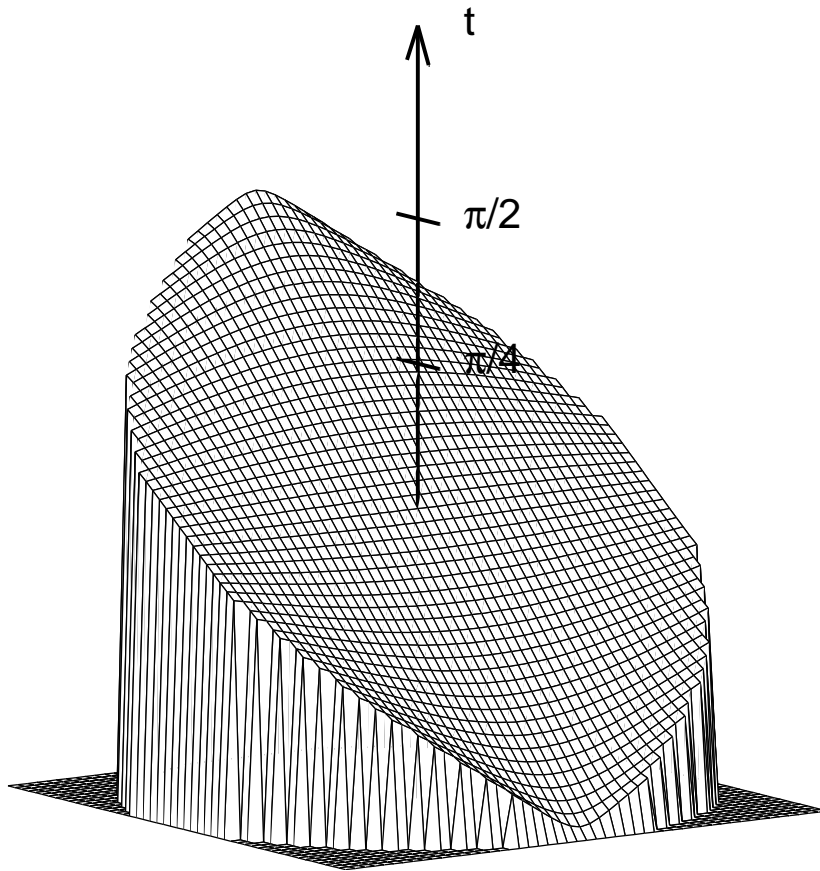## 2.3 Calculation of the Ricci Tensor

In order to see that these metrics really describe a spacetime with a negative cosmological constant, the Ricci tensor $R_{\mu\nu}$ will be calculated. The Ricci tensor is extracted from the Riemann tensor $R^\rho{}_{\mu\gamma\nu}$ by reducing two indices so that $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. Einstein's original field equation for empty space was $R_{\mu\nu} = 0$, but equally consistent with the basic argumentation is to write more generally

$$R_{\mu\nu} = \Lambda g_{\mu\nu},$$

where Λ is a constant, called the cosmological constant. Thus, if the Ricci tensor is known for empty space, the cosmological constant can be read off.

We choose to work in Poincaré coordinates, $x^\mu = (t, r, x)$. Then the Lagrangian $L = \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}$ is

$$L = \frac{b^2}{r^2}(-\dot{t}^2 + \dot{r}^2 + \dot{x}^2).$$

Knowing the Lagrangian L the acceleration F_μ is given, on the one hand, by

$$2F_\mu = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu},$$

on the other hand by

$$F_\mu = \ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta.$$

The last expression can be considered as the definition of the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$, the components of which can now be determined by identification. We get

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \Gamma^x{}_{xr} = \Gamma^x{}_{rx} = -\frac{1}{2r}$$

$$\Gamma^r{}_{rr} = \Gamma^r{}_{xx} = -\Gamma^r{}_{tt} = \frac{1}{r}$$

For convenience in the calculations we collect these elements into matrices $\mathbf{\Gamma}_\beta = \Gamma^\mu{}_{\alpha\beta}$, considering μ as row index and α as column index.

$$\mathbf{\Gamma}_t = \begin{pmatrix} 0 & -1/2r & 0 \\ -1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{\Gamma}_r = \begin{pmatrix} -1/2r & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & -1/2r \end{pmatrix}$$

$$\text{and} \qquad \mathbf{\Gamma}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/r \\ 0 & 1/2r & 0 \end{pmatrix}$$

Perform the same collection of elements for the Riemann tensor by letting $\mathbf{B}_{\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}$, where again μ is the row index and ν is the column index. The definition of the Riemann tensor,

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}{}_{\rho\nu} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\alpha}{}_{\sigma\nu} - \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\rho\nu}$$

can now be written, using matrix multiplication, as

$$\mathbf{B}_{\rho\sigma} = \partial_\rho \mathbf{\Gamma}_\sigma - \partial_\sigma \mathbf{\Gamma}_\rho + \mathbf{\Gamma}_\rho \mathbf{\Gamma}_\sigma - \mathbf{\Gamma}_\sigma \mathbf{\Gamma}_\rho.$$

Clearly, $\mathbf{B}_{\rho\sigma}$ is anti-symmetric with respect to ρ and σ. The only surviving matrices are

$$\mathbf{B}_{rt} = -\mathbf{B}_{tr} = \begin{pmatrix} 0 & 1/4r^2 & 0 \\ 1/2r^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}_{rx} = -\mathbf{B}_{xr} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2r^2 \\ 0 & 1/4r^2 & 0 \end{pmatrix}.$$

The non-zero components of the Riemann tensor can now be read off,

$$R^{t}{}_{rrt} = -R^{t}{}_{rtr} = R^{x}{}_{rrx} = -R^{x}{}_{rxr} = \frac{1}{4r^2},$$

$$R^{r}{}_{trt} = -R^{r}{}_{ttr} = R^{r}{}_{xxr} = -R^{r}{}_{xrx} = \frac{1}{2r^2}.$$

and the Ricci tensor eventually turns out to be

$$R_{\mu\nu} = \begin{pmatrix} 1/2r^2 & 0 & 0 \\ 0 & -1/2r^2 & 0 \\ 0 & 0 & -1/2r^2 \end{pmatrix} = -\frac{1}{2b^2} g_{\mu\nu}.$$

Hence, the metric expresses an empty space solution with a negative cosmological constant Λ given by

$$\Lambda = -\frac{1}{2b^2}.$$

2.4 Geodesic observers in Anti-deSitter Spacetime

We will investigate the structure of the anti-deSitter spacetime using the global coordinates. We note that the metric does not contain the time variable t explicitly, so there is no particular start or end point in time. Moreover,

as will be shown below, the proper time difference between $t = -\pi$ and $t = \pi$ is finite. We also know that the boundaries at $t = -\pi$ and $t = \pi$ are to be identified.

Now the intuition tells us that we are in trouble. Inevitably, we get time loops, with all the related problems. For time loops to work we need periodicity in everything happening in the cylinder. But so far no interactions have been introduced, and the space is empty, so the only thing which is allowed is a geodesic observer with negligible mass and no internal structure. (This of course prevents the observer from being able to observe anything, but that does not matter here.) Consequently, it is enough to investigate the possible periodicity in the geodesics.

Consider an observer fixed at the center of the cylinder, i.e. $\rho = 0$. It is easy to check that this observer moves along a geodesic. The Lagrangian, using the proper time τ as affine parameter, states that

$$-1 = L = -b^2 \, \dot{t}^{\,2},$$

since $\dot{\rho} = 0$. The proper time is obtained directly, $\tau = b^2 t$, and obviously there is a finite proper time difference between the end points of t according to this observer.

To investigate the periodicity, we derive the equation of motion for a geodesic observer moving radially in the cylinder. Since $\dot{\theta} = 0$, we have

$$-1 = L = b^2 \left(-\frac{\dot{t}^2}{\cos^2 \rho} + \frac{\dot{\rho}^2}{\cos^2 \rho} \right) \tag{1}$$

For geodesics $F_\mu = 0$, so the t-component states

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{2b^2 \dot{t}}{\cos^2 \rho} \right).$$

Integrating this and isolating the time variable yields

$$\dot{t} = \frac{k \cos^2 \rho}{b},$$

where k is a constant. Substituting this relation into equation (1), we get

$$\frac{b^2 \dot{\rho}^2}{\cos^2 \rho} = 1 - k^2 \cos^2 \rho.$$

In this equation the derivatives are with respect to the proper time τ, but what is interesting here is how ρ changes with respect to the time coordinate t. We have

$$\dot{\rho} = \frac{\mathrm{d}\rho}{\mathrm{d}t} \, \dot{t} = \frac{\mathrm{d}\rho}{\mathrm{d}t} \frac{k \cos^2 \rho}{b}.$$

The final equation of motion expressed in derivatives with respect to the time coordinate t is now

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \pm\sqrt{1 - \frac{1}{k^2\cos^2\rho}}. \tag{2}$$

This equation may be solved by making the substitution

$$x = \frac{\sin\rho}{\sqrt{1 - 1/k^2}},$$

which turns equation (2) into the form

$$\frac{1}{\sqrt{1 - x^2}}\frac{\mathrm{d}x}{\mathrm{d}t} = \pm 1$$

The left hand side is recognized as the derivative of $\arcsin x(t)$. Hence,

$$\sin\rho = \sqrt{1 - 1/k^2}\ \sin(t + t_0),$$

where t_0 is a constant of integration. Clearly, the geodesics have the desired periodicity. Figure 3 shows a cross-sectioned anti-deSitter cylinder in which are plotted four geodesics with different initial conditions. As k grows the geodesics look more zigzag, get straighter line segments with unitary slope.

2.5   Light-rays in Anti-deSitter Spacetime

An interesting feature of the anti-deSitter spacetime materializes when examining the behavior of light-rays. Here, radially outgoing rays from the middle of the cylinder are studied. Light-rays always move along null-geodesics, i.e. the Lagrangian vanish. In global coordinates we can thus write

$$0 = b^2\left(-\frac{\dot{t}^2}{\cos^2\rho} + \frac{\dot{\rho}^2}{\cos^2\rho}\right),$$

where the dot denotes derivative with respect to a suitable parameter. This yields the fairly simple differential equation $\dot{t} = \pm\dot{\rho}$, which integrates to

$$t + t_0 = \pm\rho.$$

We see that a light-ray can travel from the center out to infinity (at $\rho = \pi/2$), and back again if it is reflected somewhere arbitrarily far away, in finite proper time of an observer in the center.
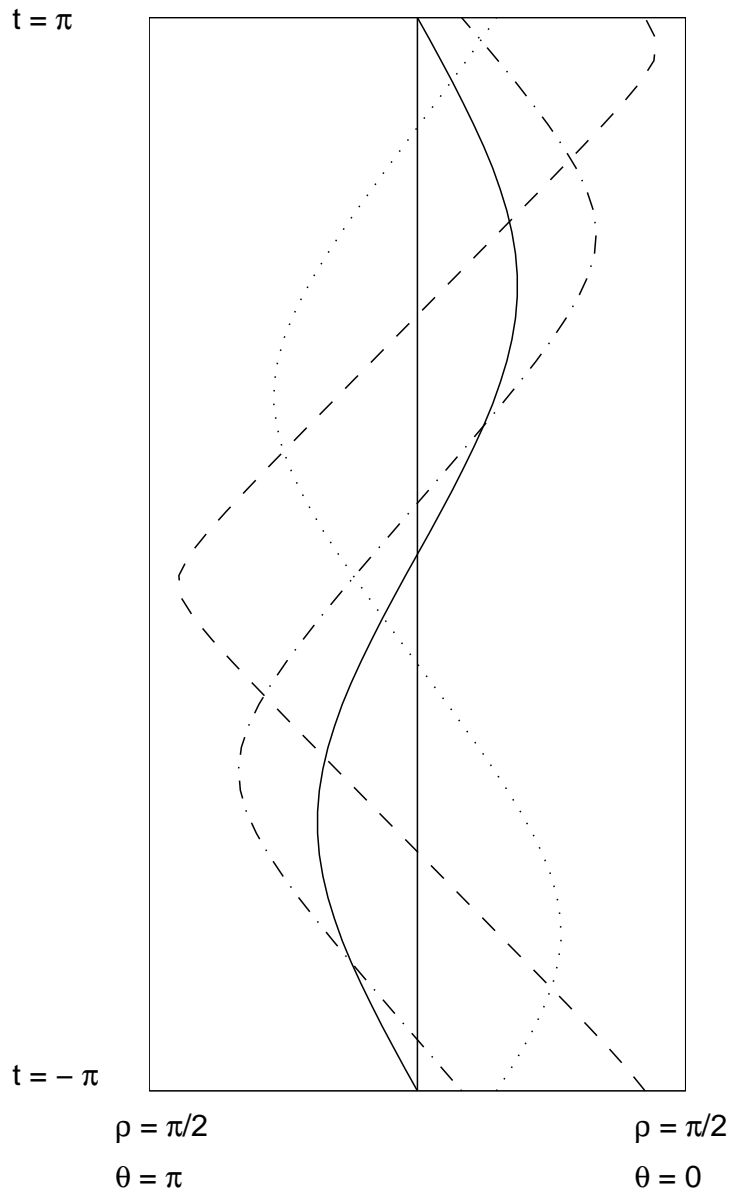
Figure 3: Cross-section of the anti-deSitter cylinder in which four geodesics are shown. They are all periodic, which is necessary as the bottom and the top of the cylinder are to be identified. As k grows the geodesic look more zigzag, get straighter line segments with unitary slope.

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3 The Holographic Principle

It is always a very fruitful situation when a discovery is made that put different parts of the physics in contradiction to each other. Many of the great steps forward being taken in the history of science are indeed forced by such a situation. The prize people have to pay is that a completely new way of looking at the world may be inevitable. A way that means abandoning ideas which were previously considered extremely natural.

As a truly surprising example of this, the development of the holographic principle is the outcome of a collision between ordinary statistical mechanics and black hole thermodynamics. The argumentation in this section will try to justify the holographic principle, without going into the very deep details. For a fuller treatment, see e.g. [7].

3.1 Violation of the Second Law

Consider a region of volume V containing N noninteracting bosons. Using Bolzmann's definition, the entropy S of this system is given by

$$S = \ln \Omega,$$

where Ω is the statistical weight, i.e. the number of possible different microscopic quantum states contributing to the same thermodynamic state. It is an obvious consequence of quantum mechanics that, if we add to the system another region with the same properties, the total number of quantum states in the new system is

$$\Omega_{\text{tot}} = \Omega^2,$$

since to each possible state for the particles in the first region there are Ω possible states for the ones in the second region. Thus, the total entropy is

$$S_{\text{tot}} = 2S.$$

Adding more regions the entropy evidently scale like

$$S = \alpha V,$$

where α is a suitable constant.

Letting the volume V of this region vary while keeping the density δ constant, there is a certain critical radius R for which, by enlarging the region even more, a black hole is formed. This critical radius is calculated straightforwardly from the Schwarzschild radius, and is given by

$$R^2 = \frac{3}{8\pi} \frac{c^2}{\delta G},$$

where c is the speed of light and G is Newton's constant. Thus, regardless of the density, it is possible to form a black hole by making the radius of the region sufficiently large.

Now, according to reliable calculations by Hawking and Bekenstein, it is possible to assign an entropy S_b to a (non-rotating and uncharged) black hole by the formula

$$S_b = \frac{\pi c^3}{\hbar G} R^2 \qquad (3)$$

A recipe for deriving this equation is given in section 5. We see that the entropy S of the region grows faster, as R increases, than the entropy S_b of a black hole of the same size. For radii satisfying

$$R > R_0 = \frac{3}{4} \frac{c^3}{\hbar \alpha G},$$

the entropy of the region exceeds the one of the equally sized black hole. If, in addition, the density is restricted to $\delta < \hbar^2 G / 3\pi$, a black hole will form at some radius $R > R_0$. But this process require a decreasing entropy, violating the second law of thermodynamics.

3.2   The Bekenstein and 't Hooft Propositions

It is of course unacceptable to have a theory in which violation of the second law is allowed. The immediate conclusion is that the maximum entropy of a region is always given by the black hole entropy formula (3). This was the original statement of Bekenstein. An attempt to extract the the underlying physics has been put forward by 't Hooft. He proposed a principle saying that everything inside a region of space can be describable by a theory, whose degrees of freedom grows proportionally to the area of the region boundary.

This is clearly in contradiction to our basic intuition. Suppose for instance that each particle possesses an n-fold degree of freedom independently of the other particles. Having N particles the total number of degrees of freedom is $f = Nn$. If we assume the density to be constant, N is proportional to the volume, and we would naively conclude that f is also proportional to the volume. The number of possible states is n^N, and the maximum entropy is

$$S = N \ln n = f \frac{\ln n}{n}.$$

However, if we follow 't Hooft and assume that the theory put an upper bound on the degrees of freedom f according to

$$\frac{\text{degrees of freedom}}{\text{boundary area}} = \frac{f}{4\pi R^2} < \frac{n}{4\ln n} \frac{c}{\hbar G},$$

then the black hole entropy formula (3) is automatically fulfilled. This means that the theory which is to describe the physics in the region, this theory possesses of the order one degree of freedom per Planck area. If this is true, we may think of the three-dimensional spatial world as being effectively two-dimensional, in a way like a holographic picture. Accordingly, this principle have been called the holographic principle.

4 Holography in Our Universe

A natural question to ask is whether the holographic principle could be fulfilled in the universe today. In cosmology there is a largest region playing any physical role, namely the region inside the particle horizon. This is the region in which the particles have had time to interact. If the entropy constraint inferred from the holographic principle holds true for this region, it holds true for any region inside it as well.

Let $R_{\mathrm{H}} = R_{\mathrm{H}}(t)$ be the coordinate at the particle horizon. The entropy S_{H} inside R_{H} must, according to the holographic principle, obey

$$S_{\mathrm{H}} < S_{\mathrm{b}} = \frac{\pi c^3}{\hbar G}(aR_{\mathrm{H}})^2. \tag{4}$$

where a is the scale factor found in the metric below, so that aR_{H} is the actual radius of the region. To check the requirement in (4) the present particle horizon has to be calculated. We assume the universe to be flat, as this seems to be a good approximation for our universe. The metric is then given by

$$ds^2 = dt^2 - a(t)\left(dr^2 + r^2 d\Omega^2\right).$$

The assumption of a flat universe at all times of course excludes the possibility of an inflationary epoch. This lack of generality might strike back on us in the end, but unfortunately an inclusion of inflation physics is beyond the scope of this text.

4.1 Calculation of the Present Particle Horizon

The particle horizon is obtained from the metric by integrating the radial coordinate along a null-geodesic starting at the big bang, up to the time t. In other words

$$R_{\mathrm{H}}(t) = \int dr = \int_0^t \frac{dt'}{a(t')}. \tag{5}$$

The evolution of a with respect to time is obtained from the Friedmann equations, which for flat space can be written

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) \tag{6}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho \tag{7}$$

Assuming a linear relation between the density and the pressure, $p = \gamma\rho$, we can readily solve for $a(t)$ and get

$$\frac{a}{a_{\mathrm{c}}} = \left(\frac{t}{t_{\mathrm{c}}}\right)^q,$$

where a_{c} and t_{c} are constants of integration, and $q = 2/3(1+\gamma)$. The standard model of the universe now assumes that the evolution, at first, was governed by the radiation density while the matter density was negligible. However, after a certain moment in time, matter has dominated the universe dynamics. This moment is called the matter-radiation equality and is denoted by t_{eq}.

In the radiation dominated era we know that $\rho = p/3$ (giving $q = 1/2$), and for the matter domination the pressure is negligible and we let $p = 0$ (giving $q = 2/3$). Now the integral in (5) is rewritten as

$$R_{\mathrm{H}}(t) = \frac{t_{\mathrm{eq}}^{1/2}}{a_{\mathrm{eq}}} \int_0^{t_{\mathrm{eq}}} \frac{\mathrm{d}t'}{t'^{1/2}} + \frac{t_{\mathrm{eq}}^{2/3}}{a_{\mathrm{eq}}} \int_{t_{\mathrm{eq}}}^t \frac{\mathrm{d}t'}{t'^{2/3}}.$$

where we have let $a_{\mathrm{c}} = a_{\mathrm{eq}}$ and $t_{\mathrm{c}} = t_{\mathrm{eq}}$. Working out the integrals, we see that

$$aR_{\mathrm{H}} = 3t - t^{2/3}t_{\mathrm{eq}}^{1/3}$$

The present age of the universe is $t_0 = 15 \cdot 10^9$ years and the radiation-matter equality occurred at $t_{\mathrm{eq}} = 1500$ years. Since t_{eq}/t_0 is small, we may neglect the contribution from the last term on the right hand side, and the horizon at $t = t_0$ is

$$aR_{\mathrm{H}} = 45 \cdot 10^9 \text{ ly}$$

Using this value in equation (4), the present upper bound on the entropy turns out to be

$$S_{\mathrm{b}} = 2 \cdot 10^{123}.$$

4.2 The Entropy at the Present

How large is the entropy of the universe at the present epoch? Since the vast majority of the particles in the universe are photons (there are 10^8 times more

photons than baryons), we can estimate the entropy by just looking at the photons. Strictly speaking, the neutrinos give a contribution of almost the same order, but for this estimate it is sufficient to only consider the photons.

Consider the universe as a cavity containing black body radiation of the temperature T. The entropy is

$$S_{\mathrm{H}} = \frac{16\pi^3}{135} \frac{k^3}{\hbar^3 c^3} (aR_{\mathrm{H}})^3 T^3. \tag{8}$$

Putting in the present values $T(t_0) = 2.7$ K and $a_0 R_{\mathrm{H}}(t_0) = 45 \cdot 10^9$ ly, we get the entropy

$$S_{\mathrm{H}} = 5 \cdot 10^{89},$$

which is well below the limit.

4.3 Future Aspects

From (4) and (8) we get the following inequality which has to be true at all times,

$$\sigma R_{\mathrm{H}} < a^2,$$

where σ is a constant and given by

$$\sigma = \frac{16\pi^2}{135} \frac{k^3 T_0^3 a_0^3 G}{\hbar^2 c^6}$$

We know that the inequality holds true at the present epoch t_0, but will this continue to be the case in the future? Let us for the moment assume that the inequality is saturated at t_0,

$$\sigma R_{\mathrm{H}}(t_0) = a_0^2.$$

We rewrite this in order to see what is needed for $\sigma R_{\mathrm{H}} < a$ as time goes on.

$$
\begin{aligned}
\sigma R_{\mathrm{H}} &= \sigma(R_{\mathrm{H}} - R_{\mathrm{H}}(t_0)) + a_0^2 \\
&= \sigma c \int_{t_0}^{t} \frac{\mathrm{d}t'}{a} + a_0^2 \\
&= \frac{\sigma c t_0^q}{a_0(1-q)} \left(t^{1-q} - t_0^{1-q} \right) + a_0^2 - a^2 + a^2 \\
&= \frac{\sigma c t_0}{a_0(1-q)} \left(\left(\frac{t}{t_0}\right)^{1-q} - 1 \right) - a_0^2 \left(\left(\frac{t}{t_0}\right)^{2q} - 1 \right) + a^2 \\
&= g(t) + a(t)^2,
\end{aligned}
$$

where

$$g(t) = \frac{\sigma c t_0}{a_0(1-q)}\left(\left(\frac{t}{t_0}\right)^{1-q} - 1\right) - a_0^2\left(\left(\frac{t}{t_0}\right)^{2q} - 1\right)$$

Thus, for $\sigma R_\mathrm{H} < a^2$ to be fulfilled at all times $t > t_0$, we need to have $g(t) \leq 0$ for $t \geq 0$ and specially $g'(t_0) \leq 0$ since we know that $g(t_0) = 0$. Obviously, this implies that

$$q \geq \frac{1}{3}, \qquad \text{and} \qquad \frac{\sigma c t_0}{2a_0^3 q} \leq 1.$$

In fact, the second constraint guarantees that $g(t) \leq 0$ for all $t \geq t_0$, so these constraints are sufficient. Of course, if the inequality $\sigma R_\mathrm{H}(t_0) < a_0^2$ is not saturated, which is certainly not the case, we can allow for a larger value of the constant $\sigma c t_0/2a_0^3 q$, but we will see that this is not needed.

That the first constraint is fulfilled follows from special relativity since $q = 2/3(1 - \gamma)$ and $|\gamma|$ must not exceed unity for the speed of sound to be smaller the speed of light. Checking the second constraint with present values, we get

$$\frac{\sigma c t_0}{2a_0^3 b} \leq 1.0 \cdot 10^{-34}.$$

Clearly, our universe will continue to obey the holographic principle for all times in the future.

4.4 A Universe Filled with Radiation

In this context it is interesting to study a flat universe consisting exclusively of photons, since there are no free parameters involved. The result has then the potential to detect fundamental inconsistencies in the theory. Here, $\gamma = 1/3$ and the scale factor is

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{1/2}.$$

The radius of the particle horizon is now

$$a R_\mathrm{H} = ac\int_0^t \frac{\mathrm{d}t'}{a(t')} = 2ct.$$

As before we want to study the relationship between S_H and S_b. It is convenient to introduce the ratio

$$\delta = \frac{S_\mathrm{H}}{S_\mathrm{b}}.$$

According to the holographic principle, $\delta < 1$ at every time. For black body radiation in the volume V and with the temperature T we have, as stated

earlier,

$$S = \frac{4\pi^2}{45} \frac{k^3}{\hbar^3 c^3} VT^3. \tag{9}$$

In this context, V is the volume of the particle horizon, a sphere with the radius aR_H. We have

$$V = \frac{4\pi}{3}(aR_\mathrm{H})^3 = \frac{32\pi}{3}(ct)^3.$$

We also know, concerning the mass density of blackbody radiation, that

$$\rho c^2 = \frac{\pi^2}{15} \frac{k^4}{\hbar^3 c^3} T^4. \tag{10}$$

We can obtain another relation for the mass density from the second Friedmann equation (7). For a flat universe we get

$$\rho c^2 = \frac{3}{8\pi} \frac{c^2}{G} \left(\frac{\dot{a}}{a}\right)^2 = \frac{3}{8\pi} \frac{c^2}{G} \frac{1}{t^2}$$

From these two relations we can extract the time dependence of the temperature, which turn out to be

$$T^4 = \frac{45\hbar^3 c^5}{8\pi^3 k^4 G} \frac{1}{t^2}$$

We can now express the entropy of the region inside the horizon as a function of time,

$$S_\mathrm{H} = \frac{4\pi^2}{45} \frac{k^3}{\hbar^3 c^3} \frac{32\pi}{3}(ct)^3 \left(\frac{45\hbar^3 c^5}{8\pi^3 k^4 G}\right)^{3/4} \frac{1}{t^{3/2}}.$$

The corresponding entropy for a black hole of the same size as the horizon, is given by

$$S_\mathrm{b} = \pi \left(2ct\right)^2 \frac{c^3}{\hbar G}.$$

As we expect, the black hole entropy S_b grows faster than the entropy of the radiation S_H. Dividing S_H with S_b we get the time dependence of δ

$$\delta = \frac{S_\mathrm{H}}{S_\mathrm{b}} = \frac{4}{3} \left(\frac{8}{45\pi}\right)^{1/4} \left(\frac{\hbar G}{c^5}\right)^{1/4} \frac{1}{\sqrt{t}}$$

where we recognize the Planck time $t_\mathrm{P} = \sqrt{\hbar G/c^5}$. After rewriting this equation slightly,

$$\delta = \frac{4}{3} \left(\frac{8}{45\pi}\right)^{1/4} \sqrt{\frac{t_\mathrm{P}}{t}} = 0.65\sqrt{\frac{t_\mathrm{P}}{t}}$$

we arrive at the conclusion that the holographic entropy bound is fulfilled at every relevant time, i.e. at times larger than the Planck time t_P. There are no adjustable parameters here, and there is no way of changing this result, ending in another conclusion. Thus we should not be surprised by finding that the entropy bound is fulfilled in a universe consisting of a mixture of radiation and matter, like the standard model of our universe, where the initial stages are dominated by radiation. Still, Fischler and Susskind find this remarkable [8].

4.5   Black Hole Formation

Let us look more closely at the just described universe, the one filled with photons. We are going to calculate the entropy of this universe just before a black hole is formed. The entropy S in terms of the density ρ is obtained from equations (9) and (10) and reads

$$S = \frac{4}{3} \left(\frac{\pi^2 c^3}{15\hbar^3} \right)^{1/4} \frac{4\pi}{3} (aR_\mathrm{H})^3 \, \rho^{3/4} \tag{11}$$

At the moment just before the black hole is formed the radius aR_H is given by the Schwarzschild radius

$$aR_\mathrm{H} = \frac{2MG}{c^2} = \frac{8\pi(aR_\mathrm{H})^3 \rho G}{3c^2},$$

from where we get

$$aR_\mathrm{H} = \sqrt{\frac{3c^2}{8\pi\rho G}}. \tag{12}$$

Substituting this radius in the equation for the entropy of the radiation (11) yields

$$S = \frac{1}{3} \left(\frac{3}{20} \right)^{1/4} \left(\frac{c^5}{\hbar G^2} \right)^{3/4} \frac{1}{\rho}^{3/4}.$$

After the black hole is formed, the entropy is given by the black hole entropy formula

$$S_\mathrm{B} = \frac{\pi c^3}{\hbar G} (aR_\mathrm{H})^2 = \frac{3}{8} \frac{c^5}{\hbar G^2} \frac{1}{\rho}$$

where equation (12) was used to eliminate aR_H. We can now write

$$S = \frac{4}{3} \left(\frac{1}{90} \right)^{1/4} S_\mathrm{B}^{3/4}.$$

Clearly, just before the black hole is being formed the actual entropy of the region is far below the entropy of the black hole just after it has been created. It might be the case that the Bekenstein limit S_b is never saturated in reality, and that 't Hooft actually over-counts the degrees of freedom [9].

5 Black Hole Entropy in Anti-deSitter

In the preceding sections we have seen problems arising when the entropy of a black hole is considered. The formula used for the black hole entropy, the Bekenstein Hawking formula, is valid in a Minkowski background, and thus applicable to our universe.

In this section, the black hole entropy is derived assuming an anti-deSitter background. If the size of the black hole is small compared to the length scale at which the curvature is important, then nothing will change. The space is then more or less minkowskian around the black hole. However, if the black hole is big or the cosmological constant is large, things will turn out to be quite different. This domain will be explored in the following.

5.1 Extending Schwarzschild to Higher Dimensions

A black hole in Minkowski background is described by the famous Schwarzschild metric,

$$\mathrm{d}s^2 = c^2(1 - a/r)\mathrm{d}t^2 - \frac{1}{1 - a/r}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 + \sin^2\theta \ \mathrm{d}\phi^2,$$

where a is the radial coordinate at the event horizon, called the Schwarzschild radius. Let us here briefly review the procedure for relating this radius to the familiar quantities M, G and c. Consider a radial non-null geodesic, for which the Lagrangian L is

$$c^2 = L = c^2(1 - a/r)\dot{t}^2 - \frac{1}{1 - a/r} \ \dot{r}^2 \tag{13}$$

where we have used the proper time τ as parameter. Since the time coordinate t is integrable, we have

$$(1 - a/r)\dot{t} = \gamma,$$

where γ is a constant of integration. Substituting this back in the Lagrangian (13) yields

$$c^2(1 - a/r) = c^2\gamma^2 - \dot{r}^2.$$

By differentiating this we get

$$\ddot{r} = -\frac{c^2 a}{2r^2}.$$

Now recall Newton's law of gravity, which for radial motion states

$$r'' = -\frac{GM}{r^2},$$

where the derivative is with respect to Newton's universal time. If we identify Newton's time with the time coordinate t, then G will not be a constant. Instead, we are lead to define G from the formula

$$\ddot{r} = -\frac{GM}{r^2},$$

so that the Schwarzschild radius can be written $a = 2GM/c^2$.

Let us stick to Newton's classical formulation for a moment. How is the law of gravity modified when changing the number of dimensions? Crucial for Newton's law as expressing a source field is Gauß' law,

$$\int_{\text{S}} \mathbf{r}'' \cdot \mathrm{d}\mathbf{S} = -4\pi GM$$

where S is a closed surface around the object and $\mathbf{r} = r\widehat{\mathbf{r}}$. The equation simply say that the GM is the source to the field $\mathbf{r}''$. Moving up to four dimensions, the closed surface around the source is a three-dimensional surface (in four dimensions an object cannot be enclosed by a two-dimensional surface). In the general case with d dimensions we want to have

$$\int_{\text{S}} \mathbf{r}_d'' \cdot \mathrm{d}\mathbf{S}_d = -A_d GM$$

where $\mathbf{r}_d = r\widehat{\mathbf{r}}_d$ and $\mathrm{d}S^i$ are d-dimensional vectors, and A_d is a constant related to the number of dimensions. The surface is now $d-1$ dimensional and scale like r^{d-1}, so we need

$$\mathbf{r}_d'' = -\frac{GM\widehat{\mathbf{r}}_d}{r^{d-1}},$$

for Gauß' law to be fulfilled. Note that the unit of G depend on the number of dimensions.

Moving back to general relativity, a reasonable extension of the Schwarzschild metric to a $(d+1)$-dimensional spacetime (d space dimensions and one time dimension) seems to be

$$\mathrm{d}s^2 = c^2(1 - (a/r)^{d-2})\mathrm{d}t^2 - \frac{1}{1 - (a/r)^{d-2}}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2,$$

where again a is the radius of the black hole and $d\Omega^2$ denotes the angular part of the metric. Following the method used for the three-dimensional case we now have

$$\ddot{r} = -(d-2)\frac{c^2 a^{d-2}}{2r^{d-1}},$$

showing that the Schwarzschild radius is given by

$$a = \left(\frac{2GM}{(d-2)c^2}\right)^{1/(d-2)}. \tag{14}$$

5.2 Including the Cosmological Constant

Naturally, the most immediate way to get the Schwarzschild metric in d dimensions would be to extend Einstein's equations to arbitrary dimension and then solve them for an empty space assuming spherical symmetry. Not surprisingly, we would arrive in the same metric if the cosmological constant is set to zero. However, if the cosmological constant is kept, the metric is slightly modified

$$ds^2 = c^2(1 - (a/r)^{d-2} + \Lambda r^2)dt^2 - \frac{1}{1 - (a/r)^{d-2} + \Lambda r^2}dr^2 - r^2 d\Omega^2.$$

There is a horizon at r_h, where r_h is the real solution to the equation

$$\Lambda r^d + r^{d-2} - a^{d-2} = 0.$$

If the black hole is big or the curvature is substantial, i.e. $r_\mathrm{h}^2 \Lambda \gg 1$, then the r^{d-2} term may be neglected, and we can approximate the horizon size

$$r_\mathrm{h} \approx \frac{a}{(a^2 \Lambda)^{1/d}}.$$

5.3 Changing to Euclidean Metric

To obtain the entropy of the black hole the method of analytic continuation will be used, where the corresponding Euclidean metric is examined, expanded at the horizon. This is done by making the substitution $t \to \tau = it$, where τ is real. The metric now reads

$$ds^2 = -c^2(1 - (a/r)^{d-2} + \Lambda r^2)d\tau^2 - \frac{1}{1 - (a/r)^{d-2} + \Lambda r^2}dr^2 - r^2 d\Omega^2.$$

Introduce a the new coordinates R and α, defined as

$$R = \frac{2a}{d(\Lambda a^2)^{1-1/d}} \left(1 - \left(\frac{a}{r}\right)^{d-2} + \Lambda r^2 \right)^{1/2}$$

$$\alpha = \frac{d(\Lambda a^2)^{1-1/d}}{2a} \, c\tau.$$

While the differential of α is trivial, we need to work a little for the differential of R, which becomes

$$dR = \frac{1}{d(\Lambda a^2)^{1-1/d}} \frac{(a/r)^{d-1} + 2\Lambda ra}{(1 - (a/r)^{d-2} + \Lambda r^2)^{1/2}} \, dr.$$

Substituting these variables in the metric, and expanding around the point $r = r_{\mathrm{h}}$, we get to the first order in R,

$$ds^2 = -R^2 d\alpha^2 - dR^2 - r_{\mathrm{h}}^2 d\Omega^2.$$

As far as the coordinates R and α are concerned, this is a Euclidean metric where α plays the role of an angle, for which reason it must be periodic. Assuming a non-conical metric α has a period of 2π. Consequently, τ is periodic, having a period given by

$$\mathrm{Period}(\tau) = \frac{4\pi a}{cd(\Lambda a^2)^{1-1/d}}$$

Within the framework of quantum field theory it is possible, but beyond the scope of this text, to show that this period can be connected to a temperature in the following way

$$\mathrm{Period}(\tau) = \frac{\hbar}{kT}.$$

The temperature at the horizon of a big black hole in an anti-deSitter space-time is thus given by

$$T = \frac{\hbar cd(\Lambda a^2)^{1-1/d}}{4\pi ak}.$$

Knowing the temperature as a function of the size of the black hole, i.e. as a function of the mass, the entropy is obtained by using the thermodynamic relation

$$dE = kT dS.$$

where E is the energy. Hence,

$$\frac{1}{c^2}\frac{\mathrm{d}S}{\mathrm{d}M} = \frac{1}{kT} = \frac{4\pi a}{\hbar c d (\Lambda a^2)^{1-1/d}}$$

Using equation (14) to get the mass dependence of a, we can write

$$\frac{\mathrm{d}S}{\mathrm{d}M} = \frac{4\pi c}{\hbar d \Lambda}\left(\frac{(d-2)c^2\Lambda}{2G}\right)^{1/d}\frac{1}{M^{1/d}}.$$

After a trivial integration we get

$$S = \frac{4\pi(d-1)c}{\hbar d^2\Lambda}\left(\frac{(d-2)c^2\Lambda}{2G}\right)^{1/d} M^{1-1/d}.$$

We observe that the mass dependence of the entropy is quite different from the minkowskian case. A similar treatment of a black hole in a minkowskian background shows that

$$S_{\mathrm{minkowski}} = \frac{4\pi c}{\hbar(d-1)}\left(\frac{2G}{(d-2)c^2}\right)^{1/(d-2)} M^{1+1/(d-2)}.$$

Now consider a volume in $d-1$ dimensions containing radiation. In three dimensions we have the familiar equations (9) and (10) describing the entropy and energy density as a function of the temperature. Apart from the factor in front these can be extended to arbitrary dimension by unit analysis, from which we get

$$S_{\mathrm{rad}} = f_d\left(\frac{1}{\hbar c}\right)^{d-1} V\,(kT)^{d-1}$$

$$\rho c^2 = \frac{(d-1)f_d}{d}\left(\frac{1}{\hbar c}\right)^{d-1}(kT)^d$$

where f_d is a dimensionless factor depending on d, and V is the volume of the $(d-1)$-dimensional region. The mass in the region is $M = \rho V$, hence

$$S_{\mathrm{rad}} = (f_d V)^{1/d}\left(\frac{dc}{(d-1)\hbar}\right)^{1-1/d} M^{1-1/d}.$$

Obviously, the mass dependence of the entropy obtained here is the same as for the large black hole in anti-deSitter spacetime. Note that the volume V is $(d-1)$-dimensional and the anti-deSitter space is d-dimensional.

5.4 Indication of Holography

As we have seen in section 2, in anti-deSitter spacetime there is a very special boundary at $\rho = \pi/2$. It can never be reached by a geodesic observer, but a light beam can travel forth and back in finite time. In section 2 we considered the two-dimensional case, but the metrics can easily be extended to higher dimensions by introducing more angular variables. Suppose the anti-deSitter spacetime has four space-like dimensions ($d = 4$) and one time direction, then the boundary at $\rho = \pi/2$ has three space-like dimensions ($d - 1 = 3$), like our world. It is possible, at least formally, to think of a world being confined to the boundary, and we may as well think of theories which only reside in the boundary, and other theories formulated in the whole spacetime.

The calculation performed above shows striking similarities in the mass dependence of the entropy, on the one hand for radiation in the boundary, on the other hand for a large black hole in the interior. This indicate that there might be a connection between field theories in the boundary, which describe radiation, and general relativity in the interior, which describes the black hole. This supposition has been confirmed in a variety of ways during the last years (e.g. [5]) and a lot of research is going on at the moment.

This connection of the theories in the boundary and in the interior is in fact an example of holography. The holographic principle states that the our world is effectively two-dimensional, and in a world of arbitrary dimension it proclaims that one of the dimensions is not used in reality. This means that it should be possible to formulate the theories describing the world in one dimension short, i.e. to formulate them in the boundary. In anti-deSitter spacetime this seems to work.

6 Summary

We have acquainted ourselves with the geometry in the anti-deSitter spacetime and seen that geodesics are periodic in time, which is also the case for the whole spacetime. The periodicity causes time loops, and this certainly generates problems if there exist processes which are independent of the space time. But since no interaction have been introduced, we do not have such processes. We can only consider geodesics. The anti-deSitter spacetime is thus self-consistent.

The motivation for proposing the holographic principle have been presented, being a result of black hole thermodynamics. The constraint on the entropy inferred from the principle was checked for our universe, and was found fulfilled. It should be noted, however, that inflation was not included

in the calculations, and it is immediately realized that inflation may destroy the calculations completely. Inflation makes the particle horizon grow exponentially in time, and the particle horizon used here is then several tens of orders of magnitude too small.

Although it has been made clear in section 4 that the holographic principle is not ruled out in the standard model of the universe, our calculations can, of course, not tell whether the holographic principle is a fundamental property of the world. For checking this, we need to reformulate the theories, allowing for just two spacelike degrees of freedom. The theorists have not yet succeeded in finding such a theory for the present spacetime. However, as we have seen examples of in the previous section, it is possible to realize the holographic principle in the anti-deSitter spacetime. There, the theory of gravity (or more precisely string theory) inside the spacetime is equivalent to a field theory in the boundary. These results may be useful since the boundary in an anti-deSitter space is similar to the spacetime of our world, allowing us to make conclusions about our field theories by examining gravity inside an anti-deSitter spacetime. But concerning the present spacetime, the mystery of holography remains.

There are still things to be done.

Acknowledgments

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